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Approximation properties on Herz spaces

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Abstract. In this paper we consider the Herz spaces $K^{\alpha}_{p,q}$, which are a natural generalization of the Lebesgue spaces L^p. We prove some approximation properties such as density of the space $C^{\infty}_{c}(\mathbb{R}^n)$, continuity of the translation, continuity of the mollification, global behavior of the convolution with smooth functions, among others.
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Propiedades de aproximación en espacios de Herz

Resumen. En este artículo consideramos los espacios de Herz $K^{\alpha}_{p,q}$, los cuales son una generalización natural de los espacios de Lebesgue L^p. Demostramos algunas propiedades de aproximación tal es como densidad del espacio $C^{\infty}_{c}(\mathbb{R}^n)$, continuidad de la traslación, continuidad de la molificación, comportamiento global de la convolución con funciones suaves, entre otras.

Palabras clave: Espacios de Herz, Molificadores, Convolución, Espacios funcionales.

1. Introduction

The Herz space $K^{\alpha}_{p,q}$ was introduced by Herz [7] as a suitable environment for the image of the Fourier transform acting on a class of Lipschitz spaces, in order to obtain a Bernstein type theorem. A characterization of the $K^{\alpha}_{p,q}$-norm in terms of L^p-norms over annuli was given in [8] (see Definition 2.1). Recently, some versions of classical spaces based on Herz spaces $K^{\alpha}_{p,q}$, such as Hardy-Herz, Sobolev-Herz and Triebel-Lizorkin-Herz spaces, have presented an increasing interest in the literature of function spaces and turned out to be a useful tool in harmonic analysis (see [1],[4],[5],[10] and references therein). We also refer the reader to [9] for results of well-posedness of the Navier-Stokes equations in weak-Herz spaces, and to [3] for results of well-posedness of the Euler equations on Besov-Herz spaces.

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In this paper we consider the Herz spaces $K^{\alpha}_{p,q}$, which are a natural generalization of the Lebesgue spaces L^p. We prove some approximation properties such as density of the spaces $C^{\infty}_c(\mathbb{R}^n)$, continuity of the translation, continuity of the mollification, global behavior of the convolution with smooth functions, among others. This results generalize the corresponding ones in L^p. They seem necessary for the study of the transport equations as done in [2] in the frame work of the L^p spaces.

This paper is organized as follows. In Section 2 we recall the definition of the Herz spaces $K^{\alpha}_{p,q}$ and some basic properties. Section 3 is devoted to the new results about approximation and global behavior of functions in Herz spaces and his convolution with smooth functions.

2. Herz spaces

This section is devoted to recall the definition of Herz spaces and establish some of their properties, which will be useful in the remainder of the paper.

For k integer and $k \geq -1$, let A_k be defined as

$$A_{-1} = B(0,1/2) \text{ and } A_k = \left\{x \in \mathbb{R}^n;\, 2^{k-1} \leq |x| < 2^k\right\} \text{ for } k \geq 0, \tag{1}$$

where $B(x_0,R) = \{x \in \mathbb{R}^n; |x-x_0| < R\}$. Then we have the disjoint decomposition $\mathbb{R}^n = \cup_{k\geq -1} A_k$.

Definition 2.1. Let $1 \leq p,q \leq \infty$ and $\alpha \in \mathbb{R}$. The Herz space $K^{\alpha}_{p,q} = K^{\alpha}_{p,q}(\mathbb{R}^n)$ is defined by

$$K^{\alpha}_{p,q} = \left\{f \in L^p_{Loc}(\mathbb{R}^n);\, \|f\|_{K^{\alpha}_{p,q}} < \infty\right\}, \tag{2}$$

where

$$\|f\|_{K^{\alpha}_{p,q}} := \begin{cases} \left(\sum_{k\geq -1} 2^{k\alpha q} \|f\|^q_{L^p(A_k)}\right)^{1/q} & \text{if } q < \infty, \\ \sup_{k\geq -1} 2^{k\alpha} \|f\|_{L^p(A_k)} & \text{if } q = \infty. \end{cases} \tag{3}$$

In the case $p = 1$, we consider $K^{\alpha}_{p,q}$ as a space of signed measures, with $\|f\|_{L^1(A_k)}$ denoting the total variation of f on A_k. The pair $(K^{\alpha}_{p,q}, \|\cdot\|_{K^{\alpha}_{p,q}})$ is a Banach space for $\alpha \in \mathbb{R}$, $1 \leq p < \infty$ and $1 \leq q \leq \infty$ (see e.g. [6]). Also, note that $K^0_{p,1} \hookrightarrow L^p = K^0_{p,p} \hookrightarrow K^0_{p,\infty}$. So, all the results presented here are a generalization of the corresponding results in L^p.

There is a version of Hölder's inequality in Herz spaces (see [9]). In fact, let $1 \leq p, p_1, p_2, q, q_1, q_2 \leq \infty$ and $\alpha, \alpha_1, \alpha_2 \in \mathbb{R}$ be such that $\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2}$, $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2}$, and $\alpha = \alpha_1 + \alpha_2$. Then,

$$\begin{aligned} \left(\sum_{k\geq -1} 2^{k\alpha q} \|fg\|^q_{L^p(A_k)}\right)^{1/q} &\leq C \left(\sum_{k\geq -1} 2^{k\alpha_1 q} \|f\|^q_{L^{p_1}(A_k)} 2^{k\alpha_2 q} \|g\|^q_{L^{p_2}(A_k)}\right)^{1/q} \\ &\leq C \|f\|_{K^{\alpha_1}_{p_1,q_1}} \|g\|_{K^{\alpha_2}_{p_2,q_2}}. \end{aligned} \tag{4}$$

In particular,

$$\|fg\|_{K^{\alpha}_{p,q}} \leq C \|f\|_{L^{\infty}(\mathbb{R}^n)} \|g\|_{K^{\alpha}_{p,q}}. \tag{5}$$

The following result provides a control in $K^{\alpha}_{p,q}$-spaces for the action of a volume preserving diffeomorphism. Recall that a diffeomorphism X is volume preserving if for all measurable set Ω, we have $\mu(\Omega) = \mu(X(\Omega))$, where μ represent the Lebesgue measure.

Lemma 2.2 ([3]). *Let $1 \leq p, q \leq \infty$ and $\alpha \geq 0$. Let $X : \mathbb{R}^n \to \mathbb{R}^n$ be a volume-preserving diffeomorphism such that, for some fixed $\omega > 0$,*

$$\left|X^{\pm 1}(y) - y\right| \leq \omega, \text{ for all } y \in \mathbb{R}^n, \tag{6}$$

where $|\cdot|$ stands for the Euclidean norm in $\mathbb{R}^n$. Then, there exists $C > 0$ such that

$$C^{-1} \|f\|_{K^{\alpha}_{p,q}} \leq \|f \circ X\|_{K^{\alpha}_{p,q}} \leq C \|f\|_{K^{\alpha}_{p,q}}, \tag{7}$$

for all $f \in K^{\alpha}_{p,q}$.

The next lemma is key in order to prove the new results in Section 3. It provides an estimate for convolution operators in Herz spaces depending on certain weighted norms of the kernel θ.

Lemma 2.3 ([3]). (Convolution). *Let $\alpha \in \mathbb{R}$ and $1 \leq p, q \leq \infty$. Let $\theta \in L^1$ be such that $M_\theta < \infty$ for some $\beta > 0$, where*

$$M_\theta = \begin{cases} max\left\{\|\theta\|_{L^1}, \left\||\cdot|^{\beta}\theta\right\|_{L^1}, \left\||\cdot|^{2\beta+\alpha}\theta\right\|_{L^1}\right\} & \text{if } \alpha \geq 0, \\ max\left\{\|\theta\|_{L^1}, \left\||\cdot|^{\beta-\alpha}\theta\right\|_{L^1}, \left\||\cdot|^{2\beta}\theta\right\|_{L^1}\right\} & \text{if } \alpha < 0. \end{cases}$$

Then, there is a constant $C > 0$ (independent of θ) such that

$$\|\theta * f\|_{K^{\alpha}_{p,q}} \leq C M_\theta \|f\|_{K^{\alpha}_{p,q}}, \text{ for all } f \in K^{\alpha}_{p,q}. \tag{8}$$

To finish this section we present a first result of density in Herz spaces.

Lemma 2.4 ([3]). *Let $1 \leq p, q \leq \infty$ and $\alpha \in \mathbb{R}$. Then, Schwartz space $\mathcal{S}$ is continuously included in $K^{\alpha}_{p,q}$. Moreover, the inclusion is dense provided that $1 \leq p, q < \infty$.*

3. New results

Now we present some new results about approximation of functions in Herz spaces and global behavior of their convolution with smooth functions. This results can be useful in order to analyze the well-posedness of some PDEs in the framework of Herz spaces.

In what follows, let $\rho \in \mathcal{S}$ such that $\int\limits_{\mathbb{R}^n} \rho(x)dx = 1$. For $\epsilon > 0$ we define $\rho_\epsilon(x) := \frac{1}{\epsilon^n}\rho\left(\frac{x}{\epsilon}\right)$, additionally, for a mensurable function $f \in \mathbb{R}^n \to \mathbb{R}$ we also define $f^{\epsilon}(x) = (\rho_\epsilon * f)(x)$.

We first show that not only $\mathcal{S}$ but also $C^{\infty}_c(\mathbb{R}^n)$ are dense in Herz spaces.

Theorem 3.1. *Let $1 \leq p, q < \infty$ and $\alpha \in \mathbb{R}$. Then, the space $C^{\infty}_c(\mathbb{R}^n)$ is dense in $K^{\alpha}_{p,q}$.*

Proof. From Lemma 2.4 it follows that $\mathcal{S}$ is dense in $K_{p,q}^{\alpha}$. Let $f \in K_{p,q}^{\alpha}$, $\epsilon > 0$ and $\phi \in \mathcal{S}$ such that $\|f-\phi\|_{K_{p,q}^{\alpha}} < \frac{\epsilon}{2}$. Now, let $N \in \mathbb{N}$ such that

$$\sum_{k \geq N+1} 2^{\alpha k q} \|\phi\|_{L^p(A_k)}^q < \left(\frac{\varepsilon}{2}\right)^q,$$

and let $\theta \in C_c^{\infty}(\mathbb{R}^n)$ be such that $0 \leq \theta \leq 1$, $\theta = 1$ in $B\left(0, 2^N\right)$ and $\theta = 0$ in $\left[B\left(0,2^N\right)\right]^c$. Defining $\psi = \theta\phi$, we have that $\psi \in C_c^{\infty}(\mathbb{R}^n)$, $\psi = \phi$ in $B\left(0,2^N\right)$ and $\theta = 0$ in $\left[B\left(0,2^N\right)\right]^c$. Moreover, it follows that

$$\begin{aligned}
\|\phi-\psi\|_{K_{p,q}^{\alpha}}^q &= \sum_{k \geq -1} 2^{\alpha k q} \|\phi - \theta\phi\|_{L^p(A_k)}^q \\
&= \sum_{k=-1}^{N} 2^{\alpha k q} \|\phi-\phi\|_{L^p(A_k)}^q + 2^{\alpha(N+1)q} \|\phi-\theta\phi\|_{L^p(A_{N+1})}^q + \sum_{k \geq N+2} 2^{\alpha k q} \|\phi\|_{L^p(A_k)}^q \\
&= 2^{\alpha(N+1)q} \|(1-\theta)\phi\|_{L^p(A_{N+1})}^q + \sum_{k \geq N+2} 2^{\alpha k q} \|\phi\|_{L^p(A_k)}^q \\
&\leq 2^{\alpha(N+1)q} \|\phi\|_{L^p(A_{N+1})}^q + \sum_{k \geq N+2} 2^{\alpha k q} \|\phi\|_{L^p(A_k)}^q = \sum_{k \geq N+1} 2^{\alpha k q} \|\phi\|_{L^p(A_k)}^q \\
&< \left(\frac{\epsilon}{2}\right)^q.
\end{aligned}$$

Thus,

$$\|f-\psi\|_{K_{p,q}^{\alpha}} \leq \|f-\theta\|_{K_{p,q}^{\alpha}} + \|\theta-\psi\|_{K_{p,q}^{\alpha}} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

Since $\epsilon > 0$ is arbitrary, we conclude the proof. ☑

Now we prove the continuity of the translation mapping in Herz spaces.

Theorem 3.2. *Let $1 \leq p, q < \infty$ and $\alpha \in \mathbb{R}$. Then, $\tau_z f \longrightarrow f$ in $K_{p,q}^{\alpha}$ as $z \longrightarrow 0$ for all $f \in K_{p,q}^{\alpha}$. Here $\tau_z f$ denotes the mapping such that $\tau_z f(x) = f(x-z)$.*

Proof. Note that, for $|z| < 1$, we have that the volume preserving diffeomorphism $X_z(x) = x - z$ trivially verifies $\|X_z - Id\|_{\infty} \leq 1$. Now, let $f \in K_{p,q}^{\alpha}$, $\epsilon > 0$ and $g \in C_c^{\infty}(\mathbb{R}^n)$ such that $\|f-g\|_{K_{p,q}^{\alpha}} < \epsilon$. Using Lemma 2.2, it follows that

$$\begin{aligned}
\|f-\tau_z f\|_{K_{p,q}^{\alpha}} &\leq \|f-g\|_{K_{p,q}^{\alpha}} + \|g-\tau_z g\|_{K_{p,q}^{\alpha}} + \|\tau_z g - \tau_z f\|_{K_{p,q}^{\alpha}} \\
&= \|f-g\|_{K_{p,q}^{\alpha}} + \|g-\tau_z g\|_{K_{p,q}^{\alpha}} + \|(g-f)\circ X_z\|_{K_{p,q}^{\alpha}} \\
&\leq \|f-g\|_{K_{p,q}^{\alpha}} + \|g-\tau_z g\|_{K_{p,q}^{\alpha}} + C\|g-f\|_{K_{p,q}^{\alpha}} \\
&\leq \epsilon + \|g-\tau_z g\|_{K_{p,q}^{\alpha}} + C\epsilon. \qquad (9)
\end{aligned}$$

Now we prove that $\|g-\tau_z g\|_{K_{p,q}^{\alpha}} \longrightarrow 0$ when $z \longrightarrow 0$. In fact, if $|z| < 1$, then supp$g$ and supp$\tau_z g$ are contained in a common compact set K. Choosing $M \in \mathbb{N}$ such that $K \subset B\left(0, 2^M\right)$, and using that $\tau_z g \longrightarrow g$ in $L^p(\mathbb{R}^n)$, we have that

$$\|g-\tau_z g\|_{K_{p,q}^{\alpha}}^q = \sum_{k=-1}^{M} 2^{\alpha k q} \|g-\tau_z g\|_{L^p(A_k)}^q \longrightarrow 0 \quad \text{as} \quad z \longrightarrow 0. \qquad (10)$$

Thus, taking lim sup as $z \longrightarrow 0$ in (9) and using (10) we obtain

$$0 \leq \limsup_{x \to \infty} \|f - \tau_z f\|_{K^{\alpha}_{p,q}} < (1+C)\epsilon;$$

since $\epsilon > 0$ is arbitrary, we obtain the result. ☑

Lemma 3.3. *Let $1 \leq p, q < \infty$, $\alpha \in \mathbb{R}$, $f \in K^{\alpha}_{p,q}$ and $\rho \in S$. Then,*

$$\|\rho_\epsilon * f\|_{K^{\alpha}_{p,q}} \leq C(\epsilon, \rho) \|f\|_{K^{\alpha}_{p,q}},$$

where $0 < C(\epsilon, \rho) \leq \tilde{C}(\rho)$ if $0 < \epsilon < 1$.

Proof. Using Lemma 2.3 we have that

$$\|\rho_\epsilon * f\|_{K^{\alpha}_{p,q}} \leq C M_{\rho_\epsilon} \|f\|_{K^{\alpha}_{p,q}},$$

where

$$M_{\rho_\epsilon} = \begin{cases} \max\left\{\|\rho_\epsilon\|_{L^1}, \left\||\cdot|^{\beta} \rho_\epsilon\right\|_{L^1}, \left\||\cdot|^{2\beta+\alpha} \rho_\epsilon\right\|_{L^1}\right\} & \text{if } \alpha \geq 0, \\ \max\left\{\|\rho_\epsilon\|_{L^1}, \left\||\cdot|^{\beta-\alpha} \rho_\epsilon\right\|_{L^1}, \left\||\cdot|^{2\beta} \rho_\epsilon\right\|_{L^1}\right\} & \text{if } \alpha < 0. \end{cases}$$

Moreover, if $0 < \epsilon < 1$ is easy to see that $M_{\rho_\epsilon} \leq M_\rho$. ☑

Now we show the continuity of the mollification of a function in Herz spaces when the mollifier is in $C^{\infty}_c(\mathbb{R}^n)$.

Theorem 3.4. *Let $f \in K^{\alpha}_{p,q}$ with $1 \leq p, q < \infty$, $\alpha \in \mathbb{R}$ and $\rho \in C^{\infty}_c(\mathbb{R}^n)$. Then, $f^\epsilon \longrightarrow f$ in $K^{\alpha}_{p,q}$ as $\epsilon \longrightarrow 0$.*

Proof. By definition we have that

$$\begin{aligned} f^\epsilon(x) - f(x) &= \int_{\mathbb{R}^n} f(x-y)\rho_\epsilon(y)dy - f(x) = \int_{\mathbb{R}^n} f(x-y)\rho_\epsilon(y)dy - f(x)\int_{\mathbb{R}^n} \rho_\epsilon(y)dy \\ &= \int_{\mathbb{R}^n} [f(x-y) - f(x)]\rho_\epsilon(y)dy = \int_{\mathbb{R}^n} [f(x-\epsilon z) - f(x)]\rho(z)dz, \end{aligned}$$

where we use the change $y = \epsilon z$. Now, using Minkowski's inequality we get

$$\|f^\epsilon - f\|_{L^p(A_k)} \leq \int_{\mathbb{R}^n} \rho(z) \|\tau_{\epsilon z} f - f\|_{L^p(A_k)} dz.$$

Thus, using again Minkowski's inequality (for sequences) we obtain

$$\begin{aligned} \|f^\epsilon - f\|_{K^{\alpha}_{p,q}} &\leq \int_{\mathbb{R}^n} \rho(z) \|\tau_{\epsilon z} f - f\|_{K^{\alpha}_{p,q}} dz \\ &\leq \int_{K} \rho(z) \|\tau_{\epsilon z} f - f\|_{K^{\alpha}_{p,q}} dz, \end{aligned} \tag{11}$$

where supp$\rho = K$. Since K is compact, we have that $|z| < C_2$ for all $z \in K$; moreover, for $0 < \epsilon \ll 1$ we also have that $|\epsilon z| < 1$ for all $z \in K$. Thus, using Lemma 2.2 we have that $\|\tau_{\epsilon z} f\|_{K^{\alpha}_{p,q}} = \|f \circ X_{\epsilon z}\|_{K^{\alpha}_{p,q}} \leq C \|f\|_{K^{\alpha}_{p,q}}$ and it follows that $\|\tau_{\epsilon z} f - f\|_{K^{\alpha}_{p,q}} \leq (1 + C) \|f\|_{K^{\alpha}_{p,q}}$. Then, using (11), Proposition 3.2 and the Dominate Convergence Theorem we obtain the result. ☑

The following is an auxiliary lemma.

Proposition 3.5. *Let $\phi \in C^{\infty}_{c}(\mathbb{R}^n)$, $1 \leq p, q \leq \infty$, $\alpha \in \mathbb{R}$ and $\rho \in \mathcal{S}$. Then, $\phi^{\epsilon} \longrightarrow \phi$ in $K^{\alpha}_{p,q}$ as $\epsilon \longrightarrow 0$.*

Proof. Let $N \in \mathbb{N}$ such that supp$\phi \subset B(0, 2^N)$, and let $M > N + 2$. For $k \geq M$ and $x \in A_k$ we have that

$$\begin{aligned}
|(\rho_\epsilon * \phi)(x)| &= \left| \int_{\mathbb{R}^n} \phi(x-y)\rho_\epsilon(y)dy \right| = \left| \int_{B(x,2^N)} \phi(x-y)\rho_\epsilon(y)dy \right| \\
&\leq \|\phi\|_{L^\infty} \int_{B(x,2^N)} \frac{1}{\epsilon^n}\rho\left(\frac{y}{\epsilon}\right) dy \\
&\leq \|\phi\|_{L^\infty} \int_{2^{k-2} \leq |y| \leq 2^{k+2}} \frac{1}{\epsilon^n}\rho\left(\frac{y}{\epsilon}\right) dy \\
&\leq \|\phi\|_{L^\infty} \int_{\frac{2^{k-2}}{\epsilon} \leq |z| \leq \frac{2^{k+2}}{\epsilon}} \rho(z)\, dz \\
&= \|\phi\|_{L^\infty} \int_{\frac{2^{k-2}}{\epsilon} \leq |z| \leq \frac{2^{k+2}}{\epsilon}} |z|^{-s} |z|^{s} \rho(z)\, dz \\
&\leq C 2^{-ks}\epsilon^s \|\phi\|_{L^\infty} \int_{\frac{2^{k-2}}{\epsilon} \leq |z| \leq \frac{2^{k+2}}{\epsilon}} |z|^{s} \rho(z)\, dz \\
&\leq C 2^{-ks}\epsilon^s \|\phi\|_{L^\infty} \left\| |\cdot|^s \rho(\cdot) \right\|_{L^1}.
\end{aligned}$$

Thus, for any $s > 0$ we get

$$\left\| |\rho_\epsilon * \phi| \right\|_{L^\infty} \leq C 2^{-ks}\epsilon^s \|\phi\|_{L^\infty} \left\| |\cdot|^s \rho(\cdot) \right\|_{L^1}. \tag{12}$$

On the other hand,

$$\|\rho_\epsilon * \phi - \phi\|^q_{K^{\alpha}_{p,q}} = \sum_{k=-1}^{M-1} 2^{\alpha k q} \|\rho_\epsilon * \phi - \phi\|^q_{L^p(A_k)} + \sum_{k \geq M} 2^{\alpha k q} \|\rho_\epsilon * \phi - \phi\|^q_{L^p(A_k)} = S_1 + S_2.$$

Is clear that $S_1 \longrightarrow 0$ as $\epsilon \longrightarrow 0$, because $\rho_\epsilon * \phi \longrightarrow \phi$ in $L^p(A_k)$. For S_2, using the

Hölder inequality, (12) and taking $s > 0$ large enough, we have

$$
\begin{aligned}
S_2 &\leq \sum_{k \geq M} 2^{\alpha k q} 2^{k \frac{n}{p} q} \left\| \rho_\epsilon * \phi - \phi \right\|_{L^\infty(A_k)}^q \leq C \sum_{k \geq M} 2^{(\alpha + \frac{n}{p}) k q} 2^{-ksq} \epsilon^{sq} \left\| \phi \right\|_{L^\infty}^q \left\| |\cdot|^s \rho(\cdot) \right\|_{L^1}^q \\
&\leq C \sum_{k \geq M} 2^{(\alpha + \frac{n}{p} - s) k q} \epsilon^{sq} \left\| \phi \right\|_{L^\infty}^q \left\| |\cdot|^s \rho(\cdot) \right\|_{L^1}^q \\
&\leq C \epsilon^{sq} \left\| \phi \right\|_{L^\infty}^q \left\| |\cdot|^s \rho(\cdot) \right\|_{L^1}^q .
\end{aligned}
$$

Then, $S_2 \longrightarrow 0$ as $\epsilon \longrightarrow 0$. The estimates for S_1 and S_2 give the result. ☑

Now we prove the continuity of the mollification of a functions in Herz spaces when the mollifier is in $\mathcal{S}$.

Theorem 3.6. *Let $f \in K_{p,q}^{\alpha}$ with $1 \leq p, q < \infty$, $\alpha \in \mathbb{R}$ and $\rho \in \mathcal{S}$. Then, $f^\epsilon \longrightarrow f$ in $K_{p,q}^{\alpha}$ as $\epsilon \longrightarrow 0$.*

Proof. Let $\delta > 0$ and $\phi \in C_c^\infty(\mathbb{R}^n)$ such that $\|\tau_{\epsilon z} f - f\|_{K_{p,q}^{\alpha}} < \delta$. For $0 < \epsilon \ll 1$, we have

$$
\begin{aligned}
\|\rho_\epsilon * f - f\|_{K_{p,q}^{\alpha}} &\leq \|\rho_\epsilon * f - \rho_\epsilon * \phi\|_{K_{p,q}^{\alpha}} + \|\rho_\epsilon * \phi - \phi\|_{K_{p,q}^{\alpha}} + \|\phi - f\|_{K_{p,q}^{\alpha}} \\
&\leq \|\rho_\epsilon (f - \phi)\|_{K_{p,q}^{\alpha}} + \|\rho_\epsilon \phi - \phi\|_{K_{p,q}^{\alpha}} + \|\phi - f\|_{K_{p,q}^{\alpha}} \\
&\leq C \|f - \phi\|_{K_{p,q}^{\alpha}} + \|\rho_\epsilon \phi - \phi\|_{K_{p,q}^{\alpha}} + \|\phi - f\|_{K_{p,q}^{\alpha}} \\
&\leq (1 + C)\delta + \|\rho_\epsilon \phi - \phi\|_{K_{p,q}^{\alpha}} .
\end{aligned}
$$

Taking $\epsilon \longrightarrow 0$ and using Lemma 3.5, we obtain

$$
\limsup_{x \to 0} \|\rho_\epsilon * f - f\|_{K_{p,q}^{\alpha}} \leq (1 + C)\delta.
$$

Since $\delta > 0$ was arbitrary, we prove the proposition. ☑

The following two results are related to the global behavior of convolutions of functions in Herz spaces with smooth functions.

Theorem 3.7. *Let $1 \leq p, q \leq \infty$, $\alpha \geq 0$, $f \in K_{p,q}^{\alpha}$ and $\psi \in C_c^\infty$. Then $f * \psi \in C^\infty(\mathbb{R}^n)$ and $\left\| D^\beta (f * \psi) \right\|_{L^\infty} \leq C(\beta) \|f\|_{K_{p,q}^{\alpha}}$ for all β multi-index.*

Proof. Note that $f * \psi \in C^\infty(\mathbb{R}^n)$ follows from the fact that $f \in L_{loc}^1$ and the classic theory of convolution. On the other hand, by definition we have

$$\begin{aligned}|(f*\psi)(x)| &\le \int\limits_{\mathbb{R}^n} |f(y)|\,|\psi(x-y)|\,dy = \sum_{k\ge -1}\int\limits_{A_k} |f(y)|\,|\psi(x-y)|\,dy\\ &\le \sum_{k\ge -1} \|f\|_{L^p(A_k)} \|\tau_x\psi\|_{L^{p'}(A_k)}\\ &\le C\sum_{k\ge -1} 2^{k\alpha}\|f\|_{L^p(A_k)} \|\tau_x\psi\|_{L^{p'}(A_k)}\\ &\le C\,\|f\|_{K^{\alpha}_{p,q}}\left(\sum_{k\ge -1}\|\tau_x\psi\|^{q'}_{L^{p'}(A_k)}\right)^{1/q'}.\end{aligned}$$

Let $N\in\mathbb{N}$ such that supp$\psi\subset B\left(0,2^N\right)$, and note that

$$\sum_{k\ge -1}\|\tau_x\psi\|^{q'}_{L^{p'}(A_k)} = \sum_{k\ge -1}\|\psi\|^{q'}_{L^{p'}((A_k-\{x\})\cap\text{supp}\psi)}.$$

Thus, if $|x|\le 2^{N+1}$, then $(A_k-\{x\})\cap\text{supp}\psi=\emptyset$ for $k\ge N+3$; therefore, for $|x|\le 2^{N+1}$ we have

$$\begin{aligned}\sum_{k\ge -1}\|\psi\|^{q'}_{L^{p'}((A_k-\{x\})\cap\text{supp}\psi)} &= \sum_{k=-1}^{N+2}\|\psi\|^{q'}_{L^{p'}((A_k-\{x\})\cap\text{supp}\psi)} \le \sum_{k=-1}^{N+2}\|\psi\|^{q'}_{L^{p'}(B(0,2^N))}\\ &= (N+3)\,\|\psi\|^{q'}_{L^{p'}(B(0,2^N))} \le C\cdot(N+3)\sum_{j=-1}^{N-1}\|\psi\|^{q'}_{L^{p'}(A_j)}\\ &\le C\cdot(N+3)\,\|\psi\|^{q'}_{K^0_{p',q'}}.\end{aligned}$$

On the other hand, if $|x|>2^{N+1}$, it follows that,

$$[(A_N-\{x\})\cup(A_{N+1}-\{x\})\cup(A_{N+2}-\{x\})]\cap B(0,2^N)\ne\emptyset,$$

and

$$(A_k-\{x\})\cap B(0,2^N)=\emptyset \text{ if } k\ne N,N+1,N+2.$$

Then,

$$\begin{aligned}\sum_{k\ge -1}\|\psi\|^{q'}_{L^{p'}((A_k-\{x\})\cap\text{supp}\psi)} &\le \sum_{k\ge -1}\|\psi\|^{q'}_{L^{p'}((A_k-\{x\})\cap B(0,2^N))}\\ &= \sum_{k=N}^{N+2}\|\psi\|^{q'}_{L^{p'}((A_k-\{x\})\cap B(0,2^N))}\\ &\le \sum_{k=N}^{N+2}\|\psi\|^{q'}_{L^{p'}(B(0,2^N))}\\ &= 3\,\|\psi\|^{q'}_{L^{p'}(B(0,2^N))} \le 3C\,\|\psi\|^{q'}_{K^0_{p',q'}}.\end{aligned}$$

Thus, for any $x \in \mathbb{R}^n$ we obtain

$$\left|\left(f * \psi\right)(x)\right| \leq C(N,p,q)\left\|\psi\right\|_{K^0_{p',q'}}\left\|f\right\|_{K^\alpha_{p,q}}.$$

Similarly, for any $x \in \mathbb{R}^n$ we have

$$\left|\partial^\beta\left(f * \psi\right)(x)\right| = \left|\left(f * \partial^\beta\psi\right)(x)\right| \leq C(N,p,q)\left\|\partial^\beta\psi\right\|_{K^0_{p',q'}}\left\|f\right\|_{K^\alpha_{p,q}}.$$ ☑

Theorem 3.8. *Let* $1 \leq p,q < \infty$*,* $\alpha \geq 0$*,* $f \in K^\alpha_{p,q}$ *and* $\psi \in C^\infty_c\left(\mathbb{R}^n\right)$*. Then* $f * \psi \in C^\infty_0\left(\mathbb{R}^n\right)$.

Proof. Let $\epsilon > 0$ and $\phi \in C^\infty_c\left(\mathbb{R}^n\right)$ such that $\left\|f - \phi\right\|_{K^\alpha_{p,q}} < \epsilon$. Then,

$$\left(f * \psi\right)(x) = \left(\left(f - \phi\right) * \psi\right)(x) + \left(\phi * \psi\right)(x).$$

Since $\phi, \psi \in C^\infty_c\left(\mathbb{R}^n\right)$, we have that $\phi * \psi \in C^\infty_c\left(\mathbb{R}^n\right)$; thus, $|x| > C = C(\epsilon)$, and we have that $\left(\phi * \psi\right)(x) = 0$. Therefore, for x large enough, it follows that

$$\left|\left(f * \psi\right)(x)\right| \leq \left|\left(\left(f - \phi\right) * \psi\right)(x)\right|.$$

Now, proceeding as in the proof of the Theorem 3.7, we get

$$\left|\left(f * \psi\right)(x)\right| \leq \left|\left(\left(f - \phi\right) * \psi\right)(x)\right| \leq C(N,p,q,\psi)\left\|f - \phi\right\|_{K^\alpha_{p,q}} \leq C(N,p,q,\psi)\epsilon.$$

Since $\epsilon > 0$ is arbitrary, we get the result. ☑

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