Continuous images of hereditarily indecomposable continua

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Abstract. The theorem proven here is that every compact metric continuum is a continuous image of some hereditarily indecomposable metric continuum.

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1. Introduction

These definitions are needed in what follows and may or may not be familiar to everyone. A continuum $X$ is a compact, connected metric space. A continuum $X$ is indecomposable provided that whenever $A$ and $B$ are proper subcontinua of $X$, $A \cup B$ is a proper subset of $X$; $X$ is hereditarily indecomposable if, and only if, every subcontinuum of $X$ is indecomposable. A map is a continuous function. A map $f$ from a continuum $X$ to a continuum $Y$ is weakly confluent provided that given any continuum $M \subseteq Y$ there exists a continuum $W \subseteq X$ such that $f(W) = M$. When $X$ is a continuum, $C(X)$ is the hyperspace of subcontinua of $X$. If $a$ and $b$ are points in $\mathbb{R}^n$ with $a \neq b$, $[a, b]$ denotes the line segment from $a$ to $b$. Let $S^n$ denote the $n$ dimensional sphere. An arc $A \subseteq S^3$ is tame if and only if there is a homeomorphism $h: S^3 \rightarrow S^3$ such that $h(A)$ is an arc of a great circle in $S^3$.

Some time later, in conversation, Rogers asked whether every continuum is a continuous image of some hereditarily indecomposable continuum. This article provides a proof that the answer to this question is also yes.

The author first announced this result in [1] but has not published it previously. It has come to my attention that in [4] this result has been extended to the non-metric case, building on the metric result.

2. Necessary Lemmas

Lemma 2.1. Let $X$ and $Y$ be continua. Then $f : X \to Y$ is weakly confluent if, and only if, the hyperspace map induced by $f$, $C(f) : C(X) \to C(Y)$, is surjective.

Proof. This is just a restatement of the definition of weakly confluent. $\blacklozenge$

Lemma 2.2. There exists a hereditarily indecomposable subcontinuum of $\mathbb{R}^4$ which separates $\mathbb{R}^4$.

Remark on proof. R. H. Bing [2] proved this not just for $n = 4$, but for every $n > 1$.

Lemma 2.3. Each homotopically essential map from a continuum $X$ to the three sphere, $S^3$, is weakly confluent.

Proof. This was essentially proven, although in a different context, by S. Mazurkiewicz in [5, Theoreme I, p. 328]. This argument gives the necessary details. Let $X$ be a continuum, and suppose $g : X \to S^3$ be a homotopically essential map. To prove that $g$ is weakly confluent, it suffices to prove that every tame arc in $S^3$ is equal to $g(M)$ for some continuum $M \subseteq X$. This follows from Lemma 2.1 because the set of tame arcs is dense in $C(S^3)$.

First, set up some machinery and notation, as follows. Let $J$ be a tame arc in $S^3$; let $D_n$ be the closed disk in the complex plane with radius $(1/n)$ centered at 0. Let $E_n$ be the corresponding open disk, and let $T_n$ be the circle $D_n \setminus E_n$. Let $C_n$ be the solid cylinder $D_n \times [0, 1]$. Since $J$ is a tame, there exists an embedding $h$ of $C_1$ into $S^3$ such that $h(\{0\} \times [0, 1]) = J$. Consider $C_n$ as a subset of $S^3$ by identifying $C_1$ with $h(C_1)$, and for each $t \in [0, 1]$ let $t$ denote the point $h(0, t) \in J$.

Let $F_n$ denote the manifold boundary of $C_n$, that is, $F_n = (D_n \times \{0, 1\}) \cup (T_n \times [0, 1])$. Note that given any $n$ and any $a, b \in J$ there is an isotopy $H : C_n \times [0, 1] \to C_n$ satisfying the following:

(i) for each $s \in [0, 1], H(J \times \{s\}) = J$;

(ii) for each $x \in F_n$ and each $t \in [0, 1], H(x, t) = x$;

(iii) for every $x \in C_n, H(x, 0) = x$; and

(iv) $H(b, 1) = a$. 
By setting \( H(x,t) = x \) for every \( x \in S^3 \setminus C_n \), and every \( t \in [0,1] \), \( H \) can be considered to be a function (hence an isotopy) from \( S^3 \times [0,1] \) to \( S^3 \).

Now, suppose \( X \) is a continuum and let \( g : X \to S^3 \) be a homotopically essential map. To prove that \( g \) is weakly confluent, it suffices to prove that there exists a continuum \( M \subseteq X \) such that \( g(M) = J \).

Proceed by contradiction; assume there is no such \( M \). Then no component of \( g^{-1}(J) \) intersects both \( g^{-1}(0) \) and \( g^{-1}(1) \). By compactness, there is a separation, \( R_0 \cup R_1 \) of \( g^{-1}(J) \) satisfying \( g^{-1}(0) \subseteq R_0 \) and \( g^{-1}(1) \subseteq R_1 \). Since \( R_0 \) and \( R_1 \) are disjoint closed sets in \( X \), there exist open subsets \( S_0 \) and \( S_1 \) of \( X \) such that \( R_0 \subseteq S_0 \) and \( R_1 \subseteq S_1 \) and \( Cl(S_0) \cap Cl(S_1) = \emptyset \). There exists \( n \) such that \( g^{-1}(Cn) \subseteq S_0 \cup S_1 \). Let \( p = \inf g(R_1) \) and let \( q = \sup g(R_0) \), and let \( a, b \in J \) be such that \( 0 < a < p \) and \( q < b < 1 \). If \( p > q \), then \( g \) is not surjective and hence not essential, so \( 0 < a < p \) and \( q < b < 1 \). Using the number \( n \) and the points \( a \) and \( b \) just chosen, let \( H : S^3 \times [0,1] \to S^3 \) be the isotopy described above. Define a homotopy \( G : X \times [0,1] \to S^3 \) by \( G(x,t) = g(x) \) if \( x \in X \setminus S_0 \) and \( G(x,t) = H(g(x),t) \) if \( x \in Cl(S_0) \). Define \( f : X \to S^3 \) by \( f(x) = G(x,1) \).

Then, note that if \( y \in J \) and \( a < y < p \), then there does not exist \( z \in X \) such that \( f(z) = y \), so \( f \) is nonsurjective. Hence, \( f \) is inessential. Since \( g \) is homotopic to \( f \), \( g \) is inessential also, a contradiction, which completes the proof. \( \Box \)

**Lemma 2.4.** A continuum \( X \subseteq R^3 \) admits a homotopically essential map onto \( S^3 \) if, and only if, \( R^4 \) \( X \) is not connected \( S^3 \).

**Remark on Proof.** This is a special case of the Borsuk separation theorem. I do not have a reference to the original proof, but a proof can be found in almost any advanced topology or algebraic topology book.

**Lemma 2.5.** Given any continuum \( Y \), there is a continuum \( X \subseteq S^3 \) that admits a continuous surjection \( f : X \to Y \).

**Proof.** Let \( Y \) be a continuum and let \( C \) and \( D \) be Cantor sets in \( R^3 \) such that \( C \) and \( D \) lie on lines skew to each other. Then, whenever \( a, p \in C \) and \( b, q \in D \), and \( a, p, b, \) and \( q \) are all different, the line segments \([a,b]\) and \([p,q]\) are disjoint. Let \( g : C \cup D \to Y \) be a map such that \( g(C) : C \to Y \) and \( g(D) : D \to Y \) are both onto. Such a \( g \) exists since a Cantor set can be mapped onto every compact metric space. Define \( X = \bigcup\{[a,b] : a \in C; b \in D \text{ and } g(a) = g(b)\} \). Then \( X \) is a continuum in \( R^3 \). For each \( x \in X \), let \([a(x),b(x)]\) be a segment in \( X \) satisfying \( a(x) \in C; b(x) \in D \), and \( x \in [a(x),b(x)] \). (This segment is unique unless \( x = a(x) \) or \( x = b(x) \).) Define \( f : X \to Y \) by \( f(x) = g(a(x)) = g(b(x)) \). It is straightforward to verify that \( f : X \to Y \) is continuous and onto. Since for any point \( p \in S^3 \), \( S^3 \setminus \{p\} \) is a copy of \( R^3 \), \( X \) can be treated as a subcontinuum of \( S^3 \). \( \Box \)

3. **Main Result**

**Theorem 3.1.** Let \( Y \) be an arbitrary continuum. There exists a hereditarily indecomposable continuum \( K \) that admits a surjective map \( f : K \to Y \).
Proof. Let $Y$ be a continuum. By Lemma 2.5, there is a continuum $T \subseteq S^3$ and an onto map $g: T \to Y$. By Lemma 2.2, there exists a hereditarily indecomposable continuum $L \subseteq R^4$ that separates $R^4$. Thus by Lemma 2.4, there is a homotopically essential map $h: L \to S^3$. By Lemma 2.3, $h$ is weakly confluent, so there exists a continuum $K \subseteq L$ such that $h(K) = T$. Let $f = g \circ (h|K)$. Then $f: K \to Y$ is the desired map; $K$ is hereditarily indecomposable since it is a subcontinuum of $L$. \hfill \Box

References


