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Ideals on countable sets: a survey with questions

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Abstract. An ideal on a set X is a collection of subsets of X closed under the operations of taking finite unions and subsets of its elements. Ideals are a very useful notion in topology and set theory and have been studied for a long time. We present a survey of results about ideals on countable sets and include many open questions.

Keywords: Ideals on countable sets, Ramsey properties, p-ideals, p^+-ideals, q^+-ideals, representation of ideals.
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Ideales sobre conjuntos numerables: un revisión con preguntas

Resumen. Un ideal sobre un conjunto X es una colección de subconjuntos de X cerrada bajo las operaciones de tomar uniones finitas y subconjuntos de sus elementos. Los ideales son una noción muy útil en topología y teoría de conjuntos y han sido estudiados desde hace mucho tiempo. Presentamos una revisión de algunos resultados sobre ideales en conjuntos numerables incluyendo preguntas abiertas sobre este tema.

Palabras clave: Ideales en conjuntos numerables, Propiedades tipo Ramsey, p-ideales, p^+-ideales, q^+-ideales, representación de ideales.

1. Introduction

An ideal on a set X is a collection of subsets of X closed under the operations of taking finite unions and subsets of its elements. Ideals are a very useful notion in topology and set theory and have been studied for a long time. We present a survey of results about ideals on countable sets and include many open questions.

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We have tried to include aspects that were not covered in the survey written by M. Hrušák [28]. We start by presenting two common forms to define ideals: based on submeasures or on collections of nowhere dense sets. A basic tool in the study of ideals are some orders to compare them: Katětov, Rudin-Keisler and Tukey order. We focus mostly on the Katětov order. The reader can consult [28], [51], [52], [53] for results on the Tukey order. One important ingredient of our presentation is that we deal mainly with definable ideals: Borel, analytic or co-analytic ideals. Another crucial aspect is the role played by combinatorial properties of ideals, a theme that has been very much studied and provides a common ground for the whole topic. Most of the work on ideals has been concentrated on tall ideals, nevertheless we include a section on Fréchet ideals (i.e., locally non tall ideals). Since the properties about ideals we are dealing with are, in one way or another, based on selection principles, we end the paper with a discussion of Borel selection principles for ideals, that is, the selection function is required to be Borel measurable.

We do not pretend to give a complete revision of this topic; in fact, the literature is vast and we have covered a small portion of it. Our purpose was to present some of the diverse ideas that have being used for studying ideals on countable sets and collect some open questions which were scattered in the literature.

2. Terminology

An ideal $\mathcal{I}$ on a set X is a collection of subsets of X such that:

(i) $\emptyset \in \mathcal{I}$ and $X \notin \mathcal{I}$.

(ii) If $A, B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$.

(iii) If $A \subseteq B$ and $B \in \mathcal{I}$, then $A \in \mathcal{I}$.

Given an ideal $\mathcal{I}$ on X, the *dual filter* of $\mathcal{I}$, denoted $\mathcal{I}^*$, is the collection of all sets $X \setminus A$ with $A \in \mathcal{I}$. We denote by $\mathcal{I}^+$ the collection of all subsets of X which do not belong to $\mathcal{I}$. Two ideals $\mathcal{I}$ and $\mathcal{J}$ on X and Y respectively are *isomorphic* if there is a bijection $f : X \to Y$ such that $E \in \mathcal{I}$ if, and only if, $f[E] \in \mathcal{J}$. Suppose X and Y are disjoint; then the *free sum* of $\mathcal{I}$ and $\mathcal{J}$, denoted by $\mathcal{I} \oplus \mathcal{J}$ is defined on $X \cup Y$ as follows: $A \in \mathcal{I} \oplus \mathcal{J}$ if $A \cap X \in \mathcal{I}$ and $A \cap Y \in \mathcal{J}$.

We denote by $2^{<\omega}$ (respectively, $\mathbb{N}^{<\omega}$) the collection of all finite binary sequences (respectively, finite sequences of natural numbers). If $x \in 2^{\mathbb{N}}$, then $x \restriction n$ is the sequence $\langle x(0), \cdots, x(n-1) \rangle$ for $n \in \mathbb{N}$.

Now we recall some combinatorial properties of ideals. We put $A \subseteq^* B$ if $A \setminus B$ is finite. An ideal $\mathcal{I}$ is a P-ideal, if for any family $E_n \in \mathcal{I}$ there is $E \in \mathcal{I}$ such that $E_n \subseteq^* E$ for all n. This is one of the most studied class of ideals.

($\mathbf{p}^+$) $\mathcal{I}$ is $\mathbf{p}^+$, if for every decreasing sequence $(A_n)_n$ of sets in $\mathcal{I}^+$, there is $A \in \mathcal{I}^+$ such that $A \subseteq^* A_n$ for all $n \in \mathbb{N}$. Following [31], we say that $\mathcal{I}$ is $\mathbf{p}^-$, if for every decreasing sequence $(A_n)_n$ of sets in $\mathcal{I}^+$ such that $A_n \setminus A_{n+1} \in \mathcal{I}$, there is $B \in \mathcal{I}^+$ such that $B \subseteq^* A_n$ for all n.

The following notion was suggested by some results in [13], [20]. Let us call a scheme a collection $\{A_s : s \in 2^{<\omega}\}$ such that $A_s = A_{s^\frown 0} \cup A_{s^\frown 1}$ and $A_{s^\frown 0} \cap A_{s^\frown 1} = \emptyset$ for all $s \in 2^{<\omega}$. An ideal is $\mathbf{wp}^+$, if for every scheme $\{A_s : s \in 2^{<\omega}\}$ with $A_\emptyset \in \mathcal{I}^+$, there is $B \in \mathcal{I}^+$ and $\alpha \in 2^{\mathbb{N}}$ such that $B \subseteq^* A_{\alpha\restriction n}$ for all n.

($\mathbf{q}^+$) $\mathcal{I}$ is $\mathbf{q}^+$, if for every $A \in \mathcal{I}^+$ and every partition $(F_n)_n$ of A into finite sets, there is $S \in \mathcal{I}^+$ such that $S \subseteq A$ and $S \cap F_n$ has at most one element for each n. Such sets S are called (partial) *selectors* for the partition. If we allow partitions with pieces in $\mathcal{I}$, we say that the ideal is *weakly selective* **ws** [31] (also called weakly Ramsey in [46]). Another natural variation is as follows: For every partition $(F_n)_n$ of a set $A \in \mathcal{I}^+$ with each piece F_n in $\mathcal{I}$, there is $S \in \mathcal{I}^+$ such that $S \subseteq A$ and $S \cap F_n$ is finite for all n. It is known that the last property is equivalent to $\mathbf{p}^-$ (see Theorem 8.2).

All spaces are assumed to be regular and T_1. A collection $\mathcal{B}$ of non empty open subsets of X is a *π-base*, if every non empty open set contains a set belonging to $\mathcal{B}$. A point x of a topological space X is called a *Fréchet point*, if for every A with $x \in \overline{A}$ there is a sequence $(x_n)_n$ in A converging to x. It is well known that filters (or dually, ideals) are viewed as spaces with only one non isolated point. We recall this basic construction. Suppose $Z = \mathbb{N} \cup \{\infty\}$ is a space such that ∞ is the only accumulation point. Then $\mathcal{F}_\infty = \{A \subseteq \mathbb{N} : \infty \in \text{int}_Z(A \cup \{\infty\})\}$ is the neighborhood filter of ∞. Conversely, given an ideal $\mathcal{I}$ over $\mathbb{N}$, we define a topology on $\mathbb{N} \cup \{\infty\}$ by declaring that each $n \in \mathbb{N}$ is isolated and $\mathcal{I}^*$ is the neighborhood filter of ∞. We denote this space by $Z(\mathcal{I})$. It is clear that the combinatorial properties of $\mathcal{I}$ and $Z(\mathcal{I})$ are the same.

For $n \in \mathbb{N}$, we denote by $X^{[n]}$ the collection of n-elements subsets of X and $X^{[\infty]}$ the collection of infinite subsets of X. The classical Ramsey theorem asserts that for every coloring $c : \mathbb{N}^{[2]} \to \{0,1\}$, there is an infinite subset X of $\mathbb{N}$ such that X is *c-homogeneous*, that is, c is constant in $X^{[2]}$. An ideal $\mathcal{I}$ is *Ramsey at* $\mathbb{N}$, when for any coloring $c : \mathbb{N}^{[2]} \to \{0,1\}$ there is a c-homogeneous set which is $\mathcal{I}$-positive, we denoted it by $\mathbb{N} \to (\mathcal{I}^+)^2_2$. If it is the case that for any coloring c and any $X \in \mathcal{I}^+$ there is a c-homogeneous set $Y \in \mathcal{I}^+$ contained in X, we shall write $\mathcal{I}^+ \to (\mathcal{I}^+)^2_2$ and call such ideal a *Ramsey* ideal. A collection $\mathcal{A}$ of subsets of a set X is *tall*, if for every infinite set $A \subseteq X$, there is an infinite set $B \subseteq A$ with $B \in \mathcal{A}$. Ramsey's theorem says that the collection of c-homogeneous sets is a tall family for any coloring c.

A general reference for all descriptive set theoretic notions used in this paper is [33]. A set is F_σ (also denoted $\mathbf{\Sigma}^0_2$) if it is equal to the union of a countable collection of closed sets. Dually, a set is G_δ (also denoted $\mathbf{\Pi}^0_2$) if it is the intersection of a countable collection of open sets. The Borel hierarchy is the collection of classess $\mathbf{\Sigma}^0_\alpha$ and $\mathbf{\Pi}^0_\alpha$ for α a countable ordinal. For instance, $\mathbf{\Pi}^0_3$ (which is also denoted by $F_{\sigma\delta}$) are the sets of the form $\bigcap_n F_n$ where each F_n is an F_σ. A subset A of a Polish space is called *analytic*, if it is a continuous image of a Polish space. Equivalently, if there is a continuous function $f : \mathbb{N}^{\mathbb{N}} \to X$ with range A, where $\mathbb{N}^{\mathbb{N}}$ is the space of irrationals. Every Borel subset of a Polish space is analytic. A subset of a Polish space is *co-analytic* if its complement is analytic. The class of analytic (resp., co-analytic) sets is denoted by $\mathbf{\Sigma}^1_1$ (resp. $\mathbf{\Pi}^1_1$).

3. Some examples

In this section we present some examples of ideals. The interested reader can consult [28], [31], [43] where he can find many more interesting examples.

The simplest ideal is Fin, the collection of all finite subsets of $\mathbb{N}$. There are two natural ideals quite related to Fin which are defined on $\mathbb{N}\times\mathbb{N}$.

$$\{\emptyset\}\times\mathsf{Fin}=\{A\subseteq\mathbb{N}\times\mathbb{N}:\ \{m\in\mathbb{N}:\ (n,m)\in A\}\in\mathsf{Fin}\text{ for all } m\in\mathbb{N}\}. \tag{1}$$

$$\mathsf{Fin}\times\{\emptyset\}=\{A\subseteq\mathbb{N}\times\mathbb{N}:\ \exists n\in\mathbb{N}\ A\subseteq\{0,1,\cdots,n\}\times\mathbb{N}\ \}. \tag{2}$$

In general, let $\mathcal{I}$ and $\mathcal{J}$ be ideals on X and Y respectively; its *Fubini product* $\mathcal{I}\times\mathcal{J}$ is an ideal on $X\times Y$ defined as follows: for $A\subseteq X\times Y$, we let $A_x=\{y\in Y\ (x,y)\in A\}$.

$$\mathcal{I}\times\mathcal{J}=\{A\subseteq X\times Y:\ \{x\in X:\ A_x\not\in\mathcal{J}\}\in\mathcal{I}\}.$$

By an abuse of notation, the ideals $\{\emptyset\}\times\mathsf{Fin}$ and $\mathsf{Fin}\times\{\emptyset\}$ are usually denoted $\emptyset\times\mathsf{Fin}$ and $\mathsf{Fin}\times\emptyset$, respectively. An ideal $\mathcal{I}$ on X is *countably generated* if there is a countable collection $\{A_n:\ n\in\mathbb{N}\}$ of subsets of X such that $E\in\mathcal{I}$ if, and only if, there is n such that $E\subseteq A_0\cup\cdots\cup A_n$. The only countably generated ideals containing all finite sets are Fin and $\mathsf{Fin}\times\{\emptyset\}$ (see Proposition 1.2.8. in [12]).

Two very important ideals on $\mathbb{Q}$ are the ideal of nowhere dense subsets of $\mathbb{Q}$ (with its usual metric topology), denoted $\mathsf{nwd}(\mathbb{Q})$, and the ideal of null sets defined as follows:

$$\mathsf{null}(\mathbb{Q})=\{A\subseteq\mathbb{Q}\cap[0,1]:\ \overline{A}\text{ has Lebesgue measure zero}\}.$$

In general, if X is a topological space, then $\mathsf{nwd}(X)$ denotes the ideal of nowhere dense subsets of X. Another very natural ideal associated to a space is defined as follows: For every point $x\in X$, let

$$\mathcal{J}_x=\{A\subseteq X:\ x\not\in\overline{A\setminus\{x\}}\}. \tag{3}$$

In fact, every ideal on X is of the form $\mathcal{J}_x$ for some topology on X.

Two ideals on $\mathbb{N}$ that have a very natural connection with number theory and real analysis are the following:

$$\mathcal{I}_{1/n}=\{A\subseteq\mathbb{N}:\ \sum_{n\in A}\frac{1}{n+1}<\infty\}$$

and

$$A\in\mathcal{I}_d\ \Leftrightarrow\ \limsup_n\frac{|A\cap\{0,1,\cdots,n-1\}|}{n}=0.$$

The ideal $\mathcal{I}_d$ consists of the *asintotic density zero sets*.

Let $CL(2^{\mathbb{N}})$ denote the collection of clopen subsests of $2^{\mathbb{N}}$ and λ the product measure on $2^{\mathbb{N}}$. Notice that $CL(2^{\mathbb{N}})$ is countable. Let

$$\Omega=\{A\in CL(2^{\mathbb{N}}):\ \lambda(A)=1/2\}.$$

Solecki's ideal $\mathcal{S}$ is the ideal on Ω generated by the following sets:

$$E_x = \{A \in \Omega : \ x \in A\},$$

where $x \in 2^{\mathbb{N}}$. Solecki introduced $\mathcal{S}$ to characterize the ideals satisfying Fatou's lemma [50].

The following ideal is called the *eventually different* ideal:

$$\mathcal{ED} = \{A \subseteq \mathbb{N} \times \mathbb{N} : (\exists n)(\forall m \geq n)(|\{(m,k) : \ k \in \mathbb{N}\} \cap A| \leq n)\}.$$

The following restriction of $\mathcal{ED}$ also plays an important role in the study of combinatorial properties of ideals:

$$\mathcal{ED}_{\mathsf{Fin}} = \mathcal{ED} \restriction \Delta,$$

where $\Delta = \{(n,m) \in \mathbb{N} \times \mathbb{N} : \ m \leq n\}$. Note that $\mathcal{ED}_{\mathsf{Fin}}$ is (up to isomorphism) the unique ideal generated by the selectors of some partition of $\mathbb{N}$ into finite sets $\{I_n : n \in \mathbb{N}\}$ such that $\limsup_n |I_n| = \infty$. As we will see later, $\mathcal{ED}_{\mathsf{Fin}}$ is critical for the q^+-property.

Let conv be the ideal generated by the range of all convergent sequences of rationals numbers, where the convergence is in $\mathbb{R}$. In other words, conv is the collection of all subsets of $\mathbb{Q}$ such that the Cantor-Bendixon derivative of its closure in $\mathbb{R}$ is finite.

Now we present a family of ideals defined by homogeneous sets for colorings. Let $c : \mathbb{N}^{[2]} \to \{0,1\}$ be a coloring. Recall that a set $H \subseteq \mathbb{N}$ is c-homogeneous if c is constant in $H^{[2]}$. The collection $hom(c)$ of all c-homogeneous sets is closed in $2^{\mathbb{N}}$. Let $\mathcal{I}_{hom(c)}$ be the ideal generated by the c-homogeneous sets.

The *infinite random graph* on $\mathbb{N}$, also known as the *Rado graph* or the *Erdös-Rényi graph* (see, e.g., [5]) can be concisely described as follows. Recall that a family $\{X_n : \ n \in \mathbb{N}\}$ of infinite subsets of $\mathbb{N}$ is *independent*, if given two disjoint finite subsets F, E of $\mathbb{N}$ the set $(\bigcap_{n \in F} X_n) \setminus (\bigcup_{n \in E} X_n)$ is infinite. Let $\{X_n : \ n \in \mathbb{N}\}$ be an independent family of subsets of $\mathbb{N}$ such that $n \in X_m$ if, and only if, $m \in X_n$, for all $m, n \in \mathbb{N}$. The random graph is then $(\mathbb{N}, E)$, where

$$E = \{\{n,m\} : m \in X_n\}.$$

The random graph is *universal* in the following sense. Given a graph $\langle \mathbb{N}, G \rangle$, there is a subset $X \subseteq \mathbb{N}$ such that $\langle \mathbb{N}, G \cong \rangle X, E \restriction X$. The *random graph ideal* $\mathcal{R}$ is the ideal on $\mathbb{N}$ generated by cliques and free sets of the random graph or, equivalently, the homogeneous sets with respect to the *random coloring* $c : [\mathbb{N}]^2 \to 2$ defined by $c(\{m,n\}) = 1$ if, and only if, $\{m,n\} \in E$.

4. Complexity of ideals

We say that a collection $\mathcal{A}$ of subsets of a countable set X is *analytic* (resp. Borel), if $\mathcal{A}$ is analytic (resp. Borel) as a subset of the cantor cubet 2^X (identifying subsets of X with characteristic functions) [33]. The set $\mathbb{N}^{[\infty]}$ of infinite subsets of $\mathbb{N}$ will be always considered with the subspace topology of $2^{\mathbb{N}}$. We say that an ideal is *analytic*, if it is an analytic as a subset of 2^X. Since the collection of finite subsets of $\mathbb{N}$ is a dense set in $2^{\mathbb{N}}$, then there are no ideals containing Fin which are closed as subsets of $2^{\mathbb{N}}$. On the other hand, if $\mathcal{I}$ is a G_δ ideal with $\mathsf{Fin} \subseteq \mathcal{I}$, then $\mathcal{I}^* = \{\mathbb{N} \setminus A : A \in \mathcal{I}\}$ is also dense G_δ (as the map $A \mapsto \mathbb{N} \setminus A$ is an homeomorphism of $2^{\mathbb{N}}$ into itself). Therefore by the Baire category theorem $\mathcal{I} \cap \mathcal{I}^* \neq \emptyset$, which says that $\mathcal{I} = \mathcal{P}(\mathbb{N})$. So, the simplest Borel ideals have complexity F_σ. They have been quite investigated as we will see. Every analytic ideal is generated by a G_δ set, i.e., there is a G_δ $G \subseteq \mathcal{I}$ such that every $A \in \mathcal{I}$ is a subset of a finite union of elements of G [64] (see also [53, Theorem 8.1]).

Most of the theory of definable ideals has been concentrated on analytic ideals. There are a few results about co-analytic ideals. The following theorem provides a very general representation of analytic ideals on spaces of continuous functions. It is an instance of the ideal defined by (3).

Theorem 4.1 (Todorčević [57, Lemma 6.53])**.** *Let $\mathcal{I}$ be an ideal over $\mathbb{N}$. The following are equivalent.*

(i) $\mathcal{I}$ is analytic.

(ii) There are continuous functions $f, f_n : \mathbb{N}^{\mathbb{N}} \to \mathbb{R}$ for $n \in \mathbb{N}$ such that f is an accumulation point of $\{f_n : n \in \mathbb{N}\}$ respect to product topology on $C_p(\mathbb{N}^{\mathbb{N}})$ and

$$E \in \mathcal{I} \text{ if, and only if, } f \notin \overline{\{f_n : n \in E\}},$$

where the closure is taken in $C_p(\mathbb{N}^{\mathbb{N}})$.

There are some well known co-analytic ideals. Let $WO(\mathbb{Q})$ be the ideal of well ordered subsets of $\mathbb{Q}$. This is a typical complete co-analytic ideal. Let $\mathcal{I}_{wf}$ be the ideal on $\mathbb{N}^{<\omega}$ generated by the well founded trees on $\mathbb{N}$, i.e., $A \subseteq \mathbb{N}^{<\omega}$ belongs to $\mathcal{I}_{wf}$, if there is a wellfounded tree T such that $A \subseteq T$. This is equivalent to say that the tree generated by A is well founded. Then $\mathcal{I}_{wf}$ is also a complete co-analytic ideal. In sections 6 and 9 we shall present another examples of co-analytic ideals (see also [16], [43]).

We do not know of any general theorem, as Theorem 4.1, for co-analytic ideals. So we state this question as follows.

Question 4.2. Is there a general representation theorem for co-analytic ideals?

5. Ideals based on submeasures

A natural and very impotant method for defining ideals is based on measures or, more generally, submeasures. In this section we present some of these ideas.

A function $\varphi : \mathcal{P}(\mathbb{N}) \to [0, \infty]$ is a *lower semicontinuous submeasure (lscsm)* if $\varphi(\emptyset) = 0$, $\varphi(A) \leq \varphi(A \cup B) \leq \varphi(A) + \varphi(B)$ and $\varphi(A) = \lim_{n\to\infty} \varphi(A \cap \{0, 1, \cdots, n\})$.

There are several ideals associated to a lscsm:

$$\begin{array}{lll} \mathsf{Fin}(\varphi) & = & \{A \subseteq \mathbb{N} : \varphi(A) < \infty\}. \\ \mathsf{Exh}(\varphi) & = & \{A \subseteq \mathbb{N} : \lim_{n\to\infty} \varphi(A \setminus \{0, 1, \ldots, n\}) = 0\}. \\ \mathsf{Sum}(\varphi) & = & \{A \subseteq \mathbb{N} : \sum_{n\in A} \varphi\{n\} < \infty\}. \end{array}$$

They satisfy the following relations:

$$\mathsf{Sum}(\varphi) \subseteq \mathsf{Exh}(\varphi) \subseteq \mathsf{Fin}(\varphi).$$

The collection of ideals that can be represented by one these three forms have been extensively investigated. The work of Farah [12] and Solecki [49] are two of the most important early works for the study of the ideals associated to submeasures.

To each divergent series $f : \mathbb{N} \to [0, +\infty)$ of possitive real numbers, we associate a measure on $\mathbb{N}$ by

$$\varphi_f(A) = \sum_{n\in A} f(n).$$

An ideal $\mathcal{I}$ is *summable* [42] if there is a divergent series f as above such that $\mathcal{I} = \mathsf{Fin}(\varphi_f)$. Notice that $\mathsf{Sum}(\varphi_f) = \mathsf{Fin}(\varphi_f)$. The usual notation for this ideal is $\mathcal{I}_f$. A typical example is the following:

$$\mathcal{I}_{1/n} = \{A \subseteq \mathbb{N} : \sum_{n\in A} \frac{1}{n+1} < \infty\}.$$

Another very natural way of defining lscsm is as follows. Let I_n be a partition of $\mathbb{N}$ into finite sets. Let ν_n be a measure on I_n (i.e., there is a function $f_n : I_n \to [0, +\infty)$ such that $\nu_n(A) = \sum_{n\in A} f_n(n)$). Let

$$\varphi(A) = \sup_n v_n(A \cap I_n).$$

Then φ is a lscsm and $Exh(\varphi)$ is called a *density ideal* [12]. The prototype is the following

Example 5.1. *Let* $\varphi_d : \mathcal{P}(\mathbb{N}) \to [0, +\infty]$ *given by*

$$\varphi_d(A) = \sup\left\{\frac{|A \cap \{0, 1, \cdots, n-1\}|}{n} : \ n \in \mathbb{N}\right\}.$$

Then $\mathcal{I}_d = \mathsf{Exh}(\varphi_d)$ *is the ideal of asintotic density zero sets. We have*

$$A \in \mathcal{I}_d \ \Leftrightarrow \ \limsup_n \frac{|A \cap \{0, 1, \cdots, n-1\}|}{n} = 0.$$

The Cantor set $\{0, 1\}^{\mathbb{N}}$ is a group with the product operation where $\{0, 1\}$ is the group $\mathbb{Z}_2$; equivalently, viewing the elements of $2^{\mathbb{N}}$ as subsets of $\mathbb{N}$, then the algebraic operation is the symmetric difference. Then $2^{\mathbb{N}}$ is actually a Polish group. Every ideal on $\mathbb{N}$ is a subgroup of $2^{\mathbb{N}}$. Since there are no G_δ ideals (containing Fin), then none of these subgroups are Polish. However, the following weaker notion has been used to study subgroups of Polish groups. We say that a subgroup G of $2^{\mathbb{N}}$ is *Polishable*, if there is a Polish group topology on G such that the Borel structure of this topology is the same as the Borel structure G inherites from $2^{\mathbb{N}}$.

The following representation of analytic P-ideals is the most fundamental result about them. It says that any P-ideal is in a sense similar to a density ideal.

Theorem 5.2 (S. Solecki [49]). *Let $\mathcal{I}$ be an analytic ideal on $\mathbb{N}$. The following are equivalent:*

(i) $\mathcal{I}$ *is a* P*-ideal.*

(ii) There is a lscsm φ *such that* $\mathcal{I} = \mathsf{Exh}(\varphi)$.

(iii) $\mathcal{I}$ *is Polishable.*

In particular, every analytic P*-ideal is* $F_{\sigma\delta}$*. Moreover,* $\mathcal{I}$ *is an* F_σ P*-ideal, if, and only if, there is a lscsm* φ *such that* $\mathcal{I} = \mathsf{Exh}(\varphi) = \mathsf{Fin}(\varphi)$.

5.1. F_σ and $F_{\sigma\delta}$ ideals

As we said, from the complexity point of view, F_σ ideals are the simplest ones. In this section we present some results about them.

A set $\mathcal{K} \subseteq \mathcal{P}(\mathbb{N})$ is *hereditary* if for every $A \in \mathcal{K}$ and $B \subseteq A$ we have that $B \in \mathcal{K}$. A family $\mathcal{K}$ of subsets of $\mathbb{N}$ is said to be *closed under finite changes* if $A \triangle F \in \mathcal{K}$ for every $A \in \mathcal{K}$ and a finite set $F \subseteq \mathbb{N}$. Given an hereditary collection $\mathcal{K}$, we denote by $\mathcal{I}_{\mathcal{K}}$ the ideal generated by $\mathcal{K}$. That is to say

$$B \in \mathcal{I}_{\mathcal{K}} \quad \Leftrightarrow \quad \exists n \in \mathbb{N}, \exists A_1, \cdots, A_n \in \mathcal{K}\ (B \subseteq A_1 \cup \cdots \cup A_n).$$

Theorem 5.3 (Mazur [42]). *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following are equivalent.*

(i) $\mathcal{I}$ *is* F_σ.

(ii) there is a hereditary closed collection $\mathcal{K}$ *of subsets of* $\mathbb{N}$ *such that* $\mathcal{I} = \mathcal{I}_{\mathcal{K}}$.

(iii) there is a lscsm φ *such that* $\mathcal{I} = \mathsf{Fin}(\varphi)$.

An important example of F_σ ideals are $\mathcal{I}_{hom(c)}$, the ideal generated by the family $hom(c)$ of homogeneous sets respect to a coloring c (see section 3). Notice that $hom(c)$ is a closed hereditary collection of subsets of $\mathbb{N}$.

The following is part of the folklore (for a proof see, e.g., [31, Lemma 3.3]).

Theorem 5.4. *Every* F_σ *ideal is* p^+.

An important question involving F_σ ideals is the following:

Question 5.5 (M. Hrušák [29]). Does every tall Borel ideal contain a tall F_σ ideal?

The previous question can be understood as asking whether an analog of the classical perfect set theorem holds for the collection of tall Borel ideals. However, the analogy is not complete, since there exists a $\mathbf{\Pi}^1_2$ ideal which does not contain any F_σ tall ideal (see [24, Theorem 4.24]).

We have seen in Theorem 5.2 that every analytic P-ideal is $F_{\sigma\delta}$. One could naturally ask whether such ideals are a countable intersection of F_σ ideals. Since this is not true in general, Farah [14] introduced a weaker property (we follow the presentation given in [30]). They called an ideal $\mathcal{I}$ *Farah* if there is a countable collection $\mathcal{K}_n$ of closed hereditary families of subsets of $\mathbb{N}$ such that

$$\mathcal{I} = \{A \subseteq \mathbb{N} : (\forall n \in \mathbb{N})(\exists m \in \mathbb{N})(A \setminus \{0, 1, \cdots, m\} \in \mathcal{K}_n)\}.$$

It is clear that every Farah ideal is $F_{\sigma\delta}$. In [14] it is shown that $\mathsf{nwd}(\mathbb{Q})$, $\mathsf{null}(\mathbb{Q})$ and every analytic P-ideal are Farah. However, there is no an F_σ ideal $\mathcal{J}$ such that $\mathsf{nwd}(\mathbb{Q}) \subseteq \mathcal{J}$.

Theorem 5.6 (M. Hrušák and D. Meza-Alcántara [30]). *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following are equivalent:*

(i) *$\mathcal{I}$ is Farah.*

(ii) *There is a sequence $\{F_n : n \in \mathbb{N}\}$ of hereditary F_σ sets closed under finite changes such that $\mathcal{I} = \bigcap_n F_n$.*

(iii) *There is a sequence $\{F_n : n \in \mathbb{N}\}$ of F_σ sets closed under finite changes such that $\mathcal{I} = \bigcap_n F_n$.*

The previous result suggests a weaking of the notion of a Farah ideal. An ideal $\mathcal{I}$ is called *weakly Farah* [30] if there is a sequence $\{F_n : n \in \mathbb{N}\}$ of hereditary F_σ sets such that $\mathcal{I} = \bigcap_n F_n$.

Question 5.7.

(i) (Farah [14]) Is every $F_{\sigma\delta}$ ideal a Farah ideal?

(ii) (M. Hrušák and D. Meza-Alcántara [30]) Is every $F_{\sigma\delta}$ ideal weakly Farah? Is every weakly Farah ideal a Farah ideal?

5.2. Summable ideals on Banach spaces

The notion of a summable ideal has been extended to ideals where the sum is calculated in a Banach space or, more generally, in a Polish abelian group [2], [63], [11]. In this section we present some results and questions about this approach.

Let G be a Polish abelian group (with additive notation) or a Banach space. Let $h : \mathbb{N} \to G$ be a sequence. We say that the series $\sum_n h(n)$ is *unconditional convergent* in G if the net $\{\sum_{n\in A} h(n) : A \in \mathsf{Fin}\}$ (where Fin is ordered by $\subseteq$) is convergent in G. This is equivalent to requiere that $\sum_n h(\pi(n))$ is convergent in G for every permutation π of $\mathbb{N}$. Let $h : \mathbb{N} \to G$ be a sequence such that $\sum_n h(n)$ does not exist. The *generalized summable ideal* $\mathcal{I}_h^G$ associated to G and h is the following [2]:

$$A \in \mathcal{I}_h^G \quad \Leftrightarrow \quad \sum_{n\in A} h(n) \text{ is unconditional convergent in } G.$$

An ideal $\mathcal{I}$ is said to be G-representable, if there is h such that $\mathcal{I} = \mathcal{I}_h^G$. Analogously, it is defined when an ideal is $\mathcal{C}$-representable for $\mathcal{C}$ a class of abelian Polish groups.

We recall that a lscsm φ is *non-pathological* [12] if $\varphi(A)$ is equal to the supremum of all $\nu(A)$ for ν a measure such that $\nu \leq \varphi$. A P-ideal is *non-pathological*, if it is equal to $Exh(\varphi)$ for some non-pathological lscsm φ.

Theorem 5.8 (Borodulin-Nadzieja, Farkas, Plebanek [2]).

(i) An ideal is $\mathbb{R}$-representable if, and only if, it is summable.

(ii) An ideal is Polish-representable if, and only if, it is an analytic P-ideal.

(iii) An analytic P-ideal is Banach-representable if, and only if, it is non-pathological.

(iv) A tall F_σ P-ideal is representable in c_0 if, and only if, it is summable.

(v) There is an F_σ tall ideal representable in l_1 which is not summable.

Question 5.9 ([2]). How to characterize analytic P-ideals which are c_0-representable?

Question 5.10 ([2]). How to characterize ideals which are l_1-representable? Are they necessarily F_σ?

6. Topological representations by nowhere dense sets

In this section we review some constructions of ideals motivated by the ideal of nowhere dense sets. We consider two different ways of presenting $\mathsf{nwd}(\mathbb{Q})$. For the first one, we see $\mathbb{Q}$ as a dense subset of $\mathbb{R}$ and we have the following representation:

$$\mathsf{nwd}(\mathbb{Q}) = \{A \subseteq \mathbb{Q} \cap [0,1] : \overline{A} \text{ is nowhere dense in } \mathbb{R}\}.$$

On the other hand, if $\mathcal{B}$ is a base for $\mathbb{Q}$ (of non empty open sets), we have

$$A \in \mathsf{nwd}(\mathbb{Q}) \quad \Leftrightarrow \quad \forall V \in \mathcal{B}, \exists W \in \mathcal{B}\ (W \subseteq V \wedge W \cap A = \emptyset).$$

We will address each of these approaches in this section.

6.1. Ideals of nowhere dense sets

A natural question is to determine for a given ideal $\mathcal{I}$ on a set X whether there is a topology τ on X such that $\mathcal{I} = \mathsf{nwd}(\tau)$. This question was studied in [6], but most of their results are for X uncountable. For X countable, in [58] are shown some general negative results (i.e., ideals for which such topology does not exist).

Since we are mostly interested in definable ideals, we will work with *analytic topologies*, i.e., topologies τ on X such that τ is analytic as a subset of 2^X (see section 4). The study of analytic topologies was initiated in [58] (see also [3], [4] ,[47], [59], [60], [61], [62]).

Let $\mathcal{I}$ be an ideal over X containing all singletons. Then the dual filter (together with $\emptyset$) is a T_1 (but not Hausdorff) topology such that its nowhere dense sets are exactly the sets in $\mathcal{I}$. The next natural question is to requiere that the topology is T_2. But before doing that, we consider the special case of Alexandroff topologies, i.e., topologies with the property that the intersection of any collection of open sets is open. Alexandroff topologies are typical T_0 but not T_1 (the discrete topology is the only T_1 Alexandroff topology) and are exactly those topologies that are closed as subsets of 2^X [58], [62].

Theorem 6.1 ([58]). *Let $\mathcal{I}$ be an ideal over a countable set X. Then $\mathcal{I} = \mathsf{nwd}(\tau)$ for some Alexandroff topology τ over X if, and only if, $\mathcal{I}$ is isomorphic to a free sum of ideals belonging to the following family: principal ideals,* Fin, $\mathsf{Fin} \times \{\emptyset\}$ *and* $\mathsf{nwd}(\mathbb{Q})$.

Now we analyze the case when τ is Hausdorff. It was known that there is no Hausdorff topology τ such that $\mathsf{nwd}(\tau) = \mathrm{FIN}$ (see [6]). In fact, there is a more general result.

Theorem 6.2 ([58]). *Let τ be an analytic Hausdorff topology over a countable set X without isolated points. Then,*

(i) $\mathsf{nwd}(\tau)$ *is* $\mathbf{\Pi}^1_2$ *and at least* $F_{\sigma\delta}$.

(ii) If there is an F_σ *set* $\mathcal{B} \subseteq 2^X$ *which is a base for* τ, *then* $\mathsf{nwd}(\tau)$ *is* $\mathbf{\Pi}^1_1$.

(iii) If (X, τ) *is a Fréchet and regular space, then it has a countable* π*-base (see Shibakov* [47, Corollary 2]*). Therefore,* $\mathsf{nwd}(\tau)$ *is* $F_{\sigma\delta}$*-complete.*

A typical example of a topology with an F_σ base is $CL(2^{\mathbb{N}})$, the collection of clopen subsets of $2^{\mathbb{N}}$ with the product topology. In [60] it is shown that $\mathsf{nwd}(CL(2^{\mathbb{N}}))$ is Borel. So, a natural question is

Question 6.3. Let (X, τ) be a countable topological space. Suppose τ has an F_σ base. Is $\mathsf{nwd}(\tau)$ Borel?

In [38] some nice examples of Hausdorff topologies on $\mathbb{N}$ are presented, whose nwd ideal has some applications in number theory.

Example 6.4 ([58]). *The following ideals are not of the form* $\mathsf{nwd}(\tau)$ *for any Hausdorff topology (on the corresponding set).*

(i) $\emptyset \times \mathcal{I}$ *for any* F_σ *ideal* $\mathcal{I}$.

(ii) The ideal of all subsets of ω^2 *with order type smaller than* ω^2.

(iii) The ideal of scattered subsets of $\mathbb{Q}$ *(i.e., subsets of* $\mathbb{Q}$ *which do not contain an order isomorphic copy of* $\mathbb{Q}$*).*

Question 6.5. Find general conditions guaranteeing that a given ideal on a countable set is of the form $\mathsf{nwd}(\tau)$ for τ a Hausdorff topology.

6.2. Topological representation

Suppose X is a Polish space and J is a σ-ideal of subsets of X. Let $D \subseteq X$ be a countable dense set. An ideal $\mathcal{I}_J$ on D is defined as follows (see [44], [36] and the references therein). Let $A \subseteq D$; then,

$$A \in \mathcal{I}_J \iff \overline{A} \in J.$$

An ideal $\mathcal{I}$ on $\mathbb{N}$ has a *topological representation* [36] if there is Polish space X, a σ-ideal J on X and a countable dense set $D \subseteq X$ such that $\mathcal{I}$ is isomorphic to $\mathcal{I}_J$. Notice that, by definition, $\mathsf{null}(\mathbb{Q})$ has a topological representation in $\mathbb{R}$. Both ideals $\mathsf{nwd}(\mathbb{Q})$ and $\mathsf{null}(\mathbb{Q})$ are tall and $F_{\sigma\delta}$. In [15] it is shown that $\mathsf{nwd}(\mathbb{Q})$ and $\mathsf{null}(\mathbb{Q})$ are not isomorphic and also that none of them is a P-ideal.

Topological representable ideals have the following interesting characterization. An ideal $\mathcal{I}$ is *countably separated* if there is a countable collection $\{X_n : n \in \mathbb{N}\}$ such that for all $A \in \mathcal{I}$ and all $B \notin \mathcal{I}$, there is n such that $A \cap X_n = \emptyset$ and $B \cap X_n \notin \mathcal{I}$. This notion was motivated by the results in [54].

Theorem 6.6 ([36, Theorem 1.1]). *Let $\mathcal{I}$ be an ideal on a countable set. The following are equivalent:*

(i) $\mathcal{I}$ has a topological representation.

(ii) $\mathcal{I}$ has a topological representation on $2^{\mathbb{N}}$ with an ideal J generated by a collection of closed nowhere subsets of $2^{\mathbb{N}}$.

(iii) $\mathcal{I}$ is tall and countably separated.

Theorem 6.7 ([36, Corollary 1.5]). *If a co-analytic ideal has a topological representation, then it is either $\mathbf{\Pi}^1_1$-complete or $F_{\sigma\delta}$-complete.*

Let us see some examples of ideals which are not topologically representable.

Example 6.8.

(i) Let $\mathcal{I}$ be an F_σ ideal on $\mathbb{N}$. We have already mentioned that $\emptyset \times \mathcal{I}$ is not of the form $\mathsf{nwd}(\tau)$ for any Hausdorff topology without isolated points (see Example 6.4). Suppose now that $\mathcal{I}$ is not tall. It is easy to verify that $\emptyset \times \mathcal{I}$ is not tall and hence it is not topologically representable.

(ii) Consider the ideal $\mathcal{I}$ of all subsets of ω^2 of order type smaller than ω^2 (see Example 6.4). Then $\mathcal{I}$ is tall but it is not countably separated. The same happens with the ideal $\mathsf{Fin} \times \mathsf{Fin}$ *(see* [36]*).*

In [37, Proposition 4.3] it was shown that every countably separated ideal is weakly selective (denoted **ws** in section 2), so the following is a natural question.

Question 6.9 ([44]). Let $\mathcal{I}$ be a tall, weakly selective $F_{\sigma\delta}$ ideal. Does $\mathcal{I}$ have a topological representation?

For the previous question, one could start with a Farah ideal instead of a $F_{\sigma\delta}$ (see section 5.1).

Since $\mathbb{Q}$ has a countable basis, then $\mathsf{nwd}(\mathbb{Q})$ is countably separated. On the other hand, $\mathsf{nwd}(CL(2^{\mathbb{N}}))$ is not weakly selective and therefore it is is not countably separated (see Example 3.9 in [3]). Thus a natural question is the following.

Question 6.10. Let (X,τ) be a countable Hausdorff space without isolated points. When is $\mathsf{nwd}(\tau)$ countably separated? When is it weakly selective?

6.3. Marczewski-Burstin representations

Let $\mathcal{F}$ be a family of non empty subsets of X. The *Marczewski ideal* associated to $\mathcal{F}$ is defined as follows (see [38] and references therein):

$$S^0(\mathcal{F}) = \{A \subseteq X : \forall V \in \mathcal{F}, \exists W \in \mathcal{F}\ (W \subseteq V \wedge W \cap A = \emptyset)\}.$$

If τ is a topology on X and $\mathcal{F}$ is a base for τ, then $S^0(\mathcal{F})$ is $\mathsf{nwd}(\tau)$. If an ideal $\mathcal{I}$ is equal to $S^0(\mathcal{F})$ for some family of non empty subsets of X, then it is said that $\mathcal{I}$ is *Marczewski-Burstin representable by* $\mathcal{F}$. When such $\mathcal{F}$ can be found countable, it is said that $\mathcal{I}$ is *Marczewski-Burstin countably representable*, which is denoted $\mathcal{MBC}$.

Example 6.11 ([38, Theorem 4.12]). $\mathsf{null}(\mathbb{Q})$ *is* $\mathcal{MBC}$.

It is clear that when $\mathcal{F}$ is an analytic collection of subsets of a countable set X, then $S^0(\mathcal{F})$ is at most $\mathbf{\Pi}^1_2$. Analogously to what happen with $\mathsf{nwd}(\tau)$ (see Theorem 6.2), if $\mathcal{F}$ is an F_σ family, then $S^0(\mathcal{F})$ is $\mathbf{\Pi}^1_1$.

Theorem 6.12 ([38, Theorem 4.4]).

(i) Let $\mathcal{I}$ be an $\mathcal{MBC}$ ideal. Then $\mathcal{I}$ is $F_{\sigma\delta}$ and countably separated.

(ii) If $\mathcal{I}$ is countably separated, then there is a $\mathcal{MBC}$ ideal $\mathcal{J}$ such that $\mathcal{I} \subseteq \mathcal{J}$.

There are two natural properties about $\mathcal{F}$ which imply that $S^0(\mathcal{F})$ is tall (see [38, Theorem 3.6]).

Question 6.13 ([38]). Let $f : \mathbb{N} \to \mathbb{Q}$ be a bijection. Let

$$\mathcal{J}_c = \{A \subseteq \mathbb{N} : \overline{f(A)} \text{ is countable}\}.$$

Then $\mathcal{J}_c$ is an ideal. Is it $\mathcal{MBC}$?

If (X, τ) is a countable topological space without isolated points and has a countable π-base, then $\mathsf{nwd}(\tau)$ is isomorphic to $\mathbb{Q}$ and clearly is $\mathcal{MBC}$. Thus we have the following.

Question 6.14. Let (X, τ) be a contable topological space without isolated points such that $\mathsf{nwd}(\tau)$ is $\mathcal{MBC}$. Is $\mathsf{nwd}(\tau)$ isomorphic to $\mathsf{nwd}(\mathbb{Q})$?

7. Ordering the collection of ideals

One of the main tools for the study of combinatorial properties of ideals are some orders (in fact, pre-order) defined on the collection of all ideals: Katětov order $\leq_K$, Rudin-Keisler order $\leq_{RK}$ and Tukey order $\leq_T$.

Let $\mathcal{I}$ and $\mathcal{J}$ be two ideals on X and Y respectively. We say that $\mathcal{I}$ is *Katětov below* $\mathcal{J}$, denoted $\mathcal{I} \leq_K \mathcal{J}$, if there is a function $f : Y \to X$ such that $f^{-1}[E] \in \mathcal{J}$ for all $E \in \mathcal{I}$. If f is finite-to-one, then we write $\mathcal{I} \leq_{KB} \mathcal{J}$ and refer to the (pre)order $\leq_{KB}$ as the *Katětov–Blass order*. We say that two ideals $\mathcal{I}$ and $\mathcal{J}$ are *Katětov equivalent*, denoted $\mathcal{I} \approx_K \mathcal{J}$, if $\mathcal{I} \leq_K \mathcal{J}$ and $\mathcal{J} \leq_K \mathcal{I}$. Let $Z(\mathcal{I})$ and $Z(\mathcal{J})$ be the corresponding spaces defined in section 2. If $f : Y \to X$ is a function we will abuse the notation and consider $f : Z(\mathcal{J}) \to Z(\mathcal{I})$ by letting $f(\infty) = \infty$. If f is a Katětov reduction between $\mathcal{I}$ and $\mathcal{J}$, then $f : Z(\mathcal{J}) \to Z(\mathcal{I})$ is clearly continuous. Conversely, if there is $f : Z(\mathcal{J}) \to Z(\mathcal{I})$ continuous with $f^{-1}(\infty) = \{\infty\}$, then $\mathcal{I} \leq_K \mathcal{J}$.

Let $(D, \leq)$ be a directed ordered set, i.e., for each $x, y \in D$, there is $z \in D$ such that $x, y \leq z$. A set $A \subseteq D$ is *bounded* if there is $x \in D$ such that $y \leq x$ for all $y \in A$. The dual notion to bounded set is that of cofinal set. A set $A \subseteq D$ is *cofinal*, if for each $x \in D$, there is $y \in A$ such that $x \leq y$. Let D and E be two directed orders. A function $f : D \to E$ is called *Tukey*, if preimages under f of sets bounded in E are bounded in D. We write $D \leq_T E$ if there is a Tukey function from D to E and we say that D is *Tukey reducible* to E.

We shall focus only on the Katětov order as it is crucial for stating some important open questions. We shall follow the works of Hrušák [29] and Meza [43] (see also [31]) which are basic references on this topic. We refer the reader to [51], [52], [53] for results on Tukey order. The Rudin-Keisler order will be defined in section 9 to state some questions.

Theorem 7.1 ([29]). *Let $\mathcal{I}$ and $\mathcal{J}$ be two ideals on $\mathbb{N}$. Then,*

(i) $\mathcal{I} \approx_K \mathsf{Fin}$ *if, and only if* $\mathcal{I}$ *is not tall.*

(ii) If $\mathcal{I} \subseteq \mathcal{J}$*, then* $\mathcal{I} \leq_K \mathcal{J}$*.*

(iii) If $X \in \mathcal{I}^+$*, then* $\mathcal{I} \leq_K \mathcal{I} \upharpoonright X$*.*

For many combinatorial properties there are ideals (usually Borel ones of a low complexity) which are critical with respect to the given property, that is, they are maximal or minimal in the Katětov order $\leq_K$ among all ideals satisfying the property. To illustrate this we present some examples (see [31] for many other similar results). A *countable splitting family* for an ideal $\mathcal{I}$ on $\mathbb{N}$ is a countable collection $\mathcal{X}$ of infinite subsets of $\mathbb{N}$ such that for every $Y \in \mathcal{I}^+$, there is $X \in \mathcal{X}$ such that $|X \cap Y| = |Y \setminus X| = \aleph_0$.

Theorem 7.2 ([31]). *Let $\mathcal{I}$ be a tall ideal on $\mathbb{N}$. Then,*

(i) $\mathbb{N} \to (\mathcal{I}^+)^2_2$ *if, and only if,* $\mathcal{R} \not\leq_K \mathcal{I}$*, where* $\mathcal{R}$ *is the random graph ideal.*

(ii) $\mathcal{I}$ *is a* q^+*-ideal if, and only if,* $\mathcal{I} \upharpoonright X \not\geq_{KB} \mathcal{ED}_{\mathsf{Fin}}$ *for every* $\mathcal{I}$*-positive set* X*.*

(iii) $\mathcal{I}$ *admits a countable splitting family if, and only if,* $\mathsf{conv} \leq_K \mathcal{I}$*.*

The following theorems show some global properties of the Katětov order.

Theorem 7.3.

(i) (H. Sakai [45]) *The family of all analytic P-ideals has a largest element with respect to* $\leq_{KB}$*, and thus also with respect to* $\leq_K$*.*

(ii) (H. Sakai [45]) *There is an analytic P-ideal* $\mathcal{J}$ *such that* $\mathcal{I} \leq_{KB} \mathcal{J}$ *for all* F_σ *ideal* $\mathcal{I}$*.*

(iii) (M. Hrušák and J. Grebík [23]) *There is no Borel tall ideal* $\leq_K$*-minimal among all Borel tall ideals.*

(iv) (Katětov, see [45]) *There is no Borel ideal which is* $\leq_K$*-maximum among all Borel ideals.*

There is a result similar to part (ii) proved by Hrušák-Meza [32] showing that there is a universal analytic P-ideal.

Next results show two very interesting dichotomies. The ideals $\mathcal{R}$, $\mathcal{ED}$, $\mathcal{Z}$ and $\mathcal{S}$ were defined in section 3.

Theorem 7.4 (M. Hrušák [29]) (Category Dichotomy). *Let* $\mathcal{I}$ *be a Borel ideal. Then either* $\mathcal{I} \leq_K \mathsf{nwd}(\mathbb{Q})$*, or there is an* $\mathcal{I}$*-positive set* X *such that* $\mathcal{ED} \leq_K \mathcal{I} \upharpoonright X$*.*

Theorem 7.5 (M. Hrušák [29]) (Measure Dichotomy). *Let $\mathcal{I}$ be an analytic P-ideal. Then either $\mathcal{I} \leq_K \mathcal{Z}$, or there is an $\mathcal{I}$-positive set X such that $\mathcal{S} \leq_K \mathcal{I} \upharpoonright X$.*

Question 7.6 (M. Hrušák [29]). Is $\mathcal{R} \leq_K \mathcal{S}$?

As we mentioned above there is no maximum among Borel ideals; however, we have the following.

Question 7.7 (H. Sakai [45]). Let $1 \leq \alpha < \omega_1$. Is there a Borel ideal $\mathcal{I}$ such that $\mathcal{J} \leq_K \mathcal{I}$ for all $\mathbf{\Sigma}^0_\alpha$ ideal $\mathcal{J}$?

The following is a fundamental problem.

Question 7.8 (M. Hrušák [31]). If $\mathcal{I}$ is a Borel tall ideal, then either there is an $\mathcal{I}$-positive set X such that $\mathcal{I} \upharpoonright X \geq_K \mathsf{conv}$, or there is an F_σ-ideal $\mathcal{J}$ containing $\mathcal{I}$.

See Theorem 8.9 for a partial answer to the previous question.

Question 7.9 (M. Hrušák [31]). Does every Borel ideal $\mathcal{I}$ satisfy that either $\mathcal{I} \geq_K \mathsf{Fin}\times\mathsf{Fin}$, or there is an $F_{\sigma\delta}$-ideal $\mathcal{J}$ such that $\mathcal{I} \subseteq \mathcal{J}$?

8. *Ramsey and convergence properties*

In this section we discuss some properties of ideals which have been motivated by properties of convergent sequences and series on $\mathbb{R}$ [17], [18], [19], [20]: Bolzano-Weierstrass, Riemann's rearrangement Theorem and convergence in functional spaces. Those properties have a natural connection with Ramsey's theorem.

We have not included the game theoretic version of Ramsey properties which is indeed a very interesting approach. We refer the reader to the work of Laflamme [39], [40].

To each ideal there is an associated notion of convergence that we describe hereunder. Let X be a topological space and $\mathcal{I}$ an ideal on $\mathbb{N}$. A sequence $(x_n)_n$ in X is $\mathcal{I}$-convergent to x, if $\{n \in \mathbb{N} : \ x_n \notin U\} \in \mathcal{I}$ for every open set U of X with $x \in U$. Notice that Fin-convergence is the usual notion of convergence of sequences.

We recall that an ideal $\mathcal{I}$ is called Ramsey at $\mathbb{N}$ when it satisfies $\mathbb{N} \to (\mathcal{I}^+)^2_2$, and it is called Ramsey when $\mathcal{I}^+ \to (\mathcal{I}^+)^2_2$ (see section 2). [1]

An ideal $\mathcal{I}$ has the *Bolzano–Weierstrass property*, denoted BW, if for any bounded sequence $\{x_n : n \in \mathbb{N}\}$ of real numbers there is an $\mathcal{I}$-positive set A such that $\{x_n : n \in A\}$ is $\mathcal{I}$-convergent. An ideal $\mathcal{I}$ has the *finite Bolzano–Weierstrass property*, denoted FinBW, if for any bounded sequence $\{x_n : n \in \mathbb{N}\}$ of real numbers there is an $\mathcal{I}$-positive set A such that $\{x_n : n \in A\}$ is convergent. An ideal $\mathcal{I}$ is Mon (or *monotone*), if for any sequence $\{x_n : n \in \mathbb{N}\}$ of real (equivalently rational) numbers there is an $\mathcal{I}$-positive set X such that $\{x_n : n \in X\}$ is monotone (possibly eventually constant). We say that $\mathcal{I}$ is hereditarely mononote, denoted $\mathsf{h\text{-}Mon}$, if $\mathcal{I} \upharpoonright A$ is Mon for all $A \notin \mathcal{I}$. Neither $\mathsf{nwd}(\mathbb{Q})$ nor $\mathcal{I}_d$ satisfy FinBW (see [19]).

[1] The reader familiar with [19], [20] should notice that what they called a Ramsey ideal (resp. h-Ramsey) we have called Ramsey at $\mathbb{N}$ (resp. Ramsey).

Theorem 8.1 ([20, Theorem 3.16]). *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following are equivalent:*

(i) $\mathcal{I} \upharpoonright A$ *is* FinBW *for every* $A \notin \mathcal{I}$.

(ii) For every collection $\{A_s : s \in 2^{<\omega}\}$ *such that* $A_\emptyset \notin \mathcal{I}$, $A_s = A_{s^\frown 0} \cup A_{s^\frown 1}$ *and* $A_{s^\frown 0} \cap A_{s^\frown 1} = \emptyset$ *for all* $s \in 2^{<\omega}$. *There is* $B \in \mathcal{I}^+$ *and* $\alpha \in 2^{\mathbb{N}}$ *such that* $B \subseteq^* A_{\alpha \upharpoonright n}$ *for all* n.

The property (ii) above was denoted $\mathbf{wp}^+$in [3], and property (i) was denoted h-FinBW in [19], [20]. The following theorem summarizes several known results in the literature (see [3] for a proof and references).

Theorem 8.2. *The following holds for ideals on a countable set.*

(i) $\mathbf{p}^+$ *implies* $\mathbf{wp}^+$.

(ii) $\mathbf{q}^+$ *and* $\mathbf{wp}^+$ *together is equivalent to Ramsey.*

(iii) Ramsey implies $\mathbf{ws}$.

(iv) $\mathbf{wp}^+$ *implies* $\mathbf{p}^-$.

(v) $\mathbf{p}^-$ *is equivalent to saying that for every partition* $(F_n)_n$ *of a set* $A \in \mathcal{I}^+$ *with each piece* F_n *in* $\mathcal{I}$, *there is* $S \in \mathcal{I}^+$ *such that* $S \subseteq A$ *and* $S \cap F_n$ *is finite for all* n.

(vi) $\mathbf{ws}$ *is equivalent to* $\mathbf{p}^-$ *together with* $\mathbf{q}^+$.

The usual proof that Fin is a FinBW ideal shows in fact more: Any p$^+$-ideal is FinBW.

Theorem 8.3 ([19, Theorem 3.4 and 4.1]). *Every ideal that can be extended to an* F_σ *ideal satisfies* FinBW.

Theorem 8.4 ([20, Fact 3.1 and Corollary 3.10]). *If an ideal is Ramsey at* $\mathbb{N}$, *then it satisfies* Mon, *and if it is* Mon, *then* FinBW *holds. Moreover, any* Mon *analytic* P*-ideal is Ramsey at* $\mathbb{N}$.

Example 8.5. $\mathcal{I}_{1/n}$ *is* FinBW *but not* Mon *(see the remark after Corollary 3.10 in* [20]*)*.

FinBW is a Ramsey theoretic property as stated in the following theorems.

Theorem 8.6 ([20, Theorem 3.11]). *Let* $\mathcal{I}$ *be a* q^+*-ideal. Then the following are equivalent:*

(i) $\mathcal{I}$ *is Ramsey at* $\mathbb{N}$.

(ii) $\mathcal{I}$ *is* Mon.

(iii) $\mathcal{I}$ is FinBW.

We have also a local version of the previous result.

Theorem 8.7 ([20, Theorem 3.16]). *Let* $\mathcal{I}$ *be an ideal. Then the following are equivalent:*

(i) $\mathcal{I}$ *is Ramsey.*

(ii) $\mathcal{I} \restriction A$ *is* Mon *for every* $A \in \mathcal{I}^+$.

(iii) $\mathcal{I} \restriction A$ *is* FinBW *for every* $A \in \mathcal{I}^+$ *and* $\mathcal{I}$ *is* q^+.

Perhaps one of the most intriguing question is the following.

Question 8.8 (Hrušák [31]). Is there a tall Ramsey Borel (or analytic) ideal?

A partial answer to Question 7.8 is the following.

Theorem 8.9 ([1, Proposition 6.5]). *Let* $\mathcal{I}$ *be an analytic* P*-ideal. The following are equivalent.*

(i) conv $\not\leq_K \mathcal{I}$.

(ii) $\mathcal{I}$ *is* FinBW.

(iii) $\mathcal{I}$ *can be extended to an* F_σ *ideal.*

We note that the equivalence of (i) and (ii) was proven in [43] (see section 5.1 in [31]), and that (ii) is equivalent to (iii) for analytic P-ideals was proven in [19, Theorem 4.2]. But the result was formally stated in [1, Proposition 6.5]). This motivates a reformulation of Question 7.8 as follows (see also Theorem 8.3).

Question 8.10 ([17, Problem 6.1]). Let $\mathcal{I}$ be a tall Borel FinBW ideal. Can $\mathcal{I}$ be extended to an F_σ ideal?

Now we turn our attention to another classical convergence property that can be reformulated in terms of ideals. A classical theorem of Riemann says that any conditional convergent series of reals numbers can be rearranged to converge to any given real number or to diverge to $+\infty$ or $-\infty$. In other words, if $(a_n)_n$ is a conditional convergent series and $r \in \mathbb{R} \cup \{+\infty, -\infty\}$, there is a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that $\sum_n a_{\pi(n)} = r$. In [18], [34] is considered a property of ideals motivated by Riemann's theorem. Let us say that an ideal $\mathcal{I}$ has the property R, if for any conditionally convergent series $\sum_n a_n$ of real numbers and for any $r \in \mathbb{R} \cup \{+\infty, -\infty\}$, there is a permutation $\pi : \mathbb{N} \to \mathbb{N}$ such that $\sum_n a_{\pi(n)} = r$ and

$$\{n \in \mathbb{N} : \pi(n) \neq n\} \in \mathcal{I}.$$

Similarly, $\mathcal{I}$ has property W, if for any conditionally convergent series of reals $\sum_n a_n$, there exists $A \in \mathcal{I}$ such that the restricted series $\sum_{n \in A} a_n$is still conditionally convergent. In [34] it is studied similar properties but for series of vectors in $\mathbb{R}^2$.

Theorem 8.11 (Filipów-Szuca [18]). *Let* $\mathcal{I}$ *be an ideal on* $\mathbb{N}$. *Then,*

(i) If $\mathcal{I}$ *has the property* R, *then it is tall.*

(ii) No summable ideal has property R.

(iii) If $\mathcal{I}$ *is not* BW, *then it has property* R.

For instance, since $\mathcal{I}_d$ is not BW, then it has property R.

Theorem 8.12 (Filipów-Szuca [18, Theorem 3.3]). *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following statements are equivalent.*

(i) $\mathcal{I}$ has the property R.

(ii) There is no a summable ideal $\mathcal{J}$ such that $\mathcal{I} \subseteq \mathcal{J}$.

(iii) $\mathcal{I}$ has the property W.

Question 8.13 (Klinga-Nowik [34]). Suppose that (i) $\mathcal{I}$ has the R property; (ii) $\sum_n a_n$ is a conditionally convergent series of reals; (iii) $\sum_n b_n$ is divergent and all b_n are positive reals. Does there exist $W \in \mathcal{I}$ such that $\sum_{n\in W} a_n$ is conditionally convergent and $\sum_{n\in W} b_n = \infty$?

Now we will look at some convergence properties on spaces of continuous functions. We start with the classical Arzelà-Ascoli's theorem characterizing compactness on the pointwise topology.

Theorem 8.14 ([17, Theorem 3.1]) (Ideal Version of Arzelà-Ascoli Theorem). *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. The following conditions are equivalent.*

(i) $\mathcal{I}$ is a BW *(*FinBW*, respectively).*

(ii) For every uniformly bounded and equicontinuous sequence $(f_n)_{n\in\mathbb{N}}$ of continuous real-valued functions defined on $[0,1]$, there exists $A \in \mathcal{I}^+$ such that $(f_n)_{n\in A}$ is uniformly $\mathcal{I}$-convergent (uniformly convergent, respectively).

Now we present an ideal version of the classical Helly's selection theorem in the space of monotone functions on the unit interval.

Theorem 8.15 ([17, Theorem 5.8]) (Ideal Version of Helly's Theorem). *Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. Suppose that $\mathcal{I}$ can be extended to an F_σ ideal. Then for every sequence $(f_n)_{n\in\mathbb{N}}$ of uniformly bounded monotone real-valued functions defined on $\mathbb{R}$ there is $A \in \mathcal{I}^+$ such that the subsequence $(f_n)_{n\in A}$ is pointwise convergent.*

We recall that, by Theorem 8.3, any ideal that can be extended to an F_σ ideal satisfies FinBW; thus, we have the following natural question.

Question 8.16 ([17, Problem 5.10]). Let $\mathcal{I}$ be an ideal on $\mathbb{N}$. Are the following conditions equivalent?

(i) $\mathcal{I}$ is an BW ideal (FinBW ideal, respectively).

(ii) For every uniformly bounded monotone real-valued functions $(f_n)_{n\in\mathbb{N}}$ defined on $\mathbb{R}$, there is $A \in \mathcal{I}^+$ such that the subsequence $(f_n)_{n\in A}$ is pointwise $\mathcal{I}$-convergent (pointwise convergent, respectively).

A summary of implications among some of the combinatorial properties studied is as follows. We abbreviate countably generated and countably separated by ω-gen and ω-sep, respectively. An ideal is Fréchet if it is locally non tall (they will be discussed in the next section).

$$
\begin{array}{ccccccccc}
 & & & & & & \omega\text{-gen} & & \\
 & & & & & \swarrow & & \searrow & \\
 & & & & \text{Fréchet} & & & & F_\sigma \\
 & & & & \downarrow & & & & \downarrow \\
\omega\text{-sep} & & & & \text{Ramsey} & & & & \mathbf{p}^+ \\
 & \searrow & & \swarrow & & \searrow & & \swarrow & \\
 & & \mathbf{ws} & & & & \mathbf{wp}^+ & & \\
 & \swarrow & & \searrow & & \swarrow & & & \\
\mathbf{q}^+ & & & & \mathbf{p}^- & & & &
\end{array}
$$

9. Fréchet ideals

Many of the results presented so far were about tall ideals. In this section we study Fréchet ideals, a very important class of non tall ideals. This notion has a topological motivation but it can be expressed also as a combinatorial notion. Recall that to each ideal $\mathcal{I}$ on a set X is associate a topological space $Z(\mathcal{I})$ on $X \cup \{\infty\}$ (see section 2). We say that $\mathcal{I}$ is *Fréchet* if $Z(\mathcal{I})$ is a Fréchet space. Notice that for $E \subseteq X$, we have

$$\infty \in \overline{E} \text{ if, and only if, } E \notin \mathcal{I}.$$

It is easy to verify that $\mathcal{I}$ is Fréchet if, and only if, for every $A \notin \mathcal{I}$ there is an infinite $B \subseteq A$ such that every infinite subset of B is not in $\mathcal{I}$; that is to say, $\mathcal{I} \restriction A$ is not tall for every $A \notin \mathcal{I}$. In other words, an ideal is Fréchet if it is locally non tall.

Given a family $\mathcal{A}$ of infinite subsets of X, we define the *orthogonal* of $\mathcal{A}$ as follows [54]:

$$\mathcal{A}^\perp = \{E \subseteq X : \ E \cap A \text{ is finite for all } A \in \mathcal{A}\}.$$

Notice that $\mathcal{A}^\perp$ is an ideal. If $\mathcal{A}$ is an analytic family, then $\mathcal{A}^\perp$ is co-analytic.

We denote by $\mathcal{I}(\mathcal{A})$ the ideal generated by $\mathcal{A}$, that is to say,

$$E \in \mathcal{I}(\mathcal{A}) \ \Leftrightarrow \ E \subseteq A_1 \cup \cdots \cup A_n \text{ for some } A_1, \cdots, A_n \in \mathcal{A}.$$

Example 9.1. *Let $A_n = \{n\} \times \mathbb{N}$ for $n \in \mathbb{N}$ and $\mathcal{A} = \{A_n : \ n \in \mathbb{N}\}$. Then $\mathcal{I}(\mathcal{A}) = \mathsf{Fin} \times \{\emptyset\}$.*

The following fact shows the importance of $\perp$ to study Fréchet spaces.

Theorem 9.2. *Let $\mathcal{I}$ be an ideal on X.*

(i) An infinite set $E \subseteq X$ is a convergent sequence to ∞ in $Z(\mathcal{I})$ if, and only if, $E \in \mathcal{I}^\perp$.

(i) $\mathcal{I}$ is Fréchet if, and only if, $\mathcal{I} = (\mathcal{I}^\perp)^\perp$.

Example 9.3. $(\mathsf{Fin} \times \{\emptyset\})^\perp = \{\emptyset\} \times \mathsf{Fin}$ *and* $(\{\emptyset\} \times \mathsf{Fin})^\perp = \mathsf{Fin} \times \{\emptyset\}$. *In particular,* $\mathsf{Fin} \times \{\emptyset\}$ *and* $\{\emptyset\} \times \mathsf{Fin}$ *are Fréchet ideals.*

Notice also that $\mathcal{A}^\perp = ((\mathcal{A}^\perp)^\perp)^\perp$. In other words, $\mathcal{A}^\perp$ is a Fréchet ideal for any family of sets $\mathcal{A}$.

A family $\mathcal{A}$ of subsets of X is *almost disjoint* if $A \cap B$ is finite for all $A, B \in \mathcal{A}$ with $A \neq B$. Typical examples of almost disjoint families are the following.

Example 9.4.

(i) For each irrational number r, pick a sequence $A_r = \{x_n^r : n \in \mathbb{N}\}$ of rationals numbers converging to r. Let $\mathcal{A}$ be the collection of all A_r with $r \in \mathbb{R} \setminus \mathbb{Q}$. Then $\mathcal{A}$ is an almost disjoint family of size $2^{\aleph_0}$.

(ii) Recall that $2^{<\omega}$ denotes the collection of all finite binary sequences. For each $x \in 2^{\mathbb{N}}$, let $A_x = \{x \restriction n : n \in \mathbb{N}\}$. Then $\{A_x : x \in 2^{\mathbb{N}}\}$ is an almost disjoint family.

As we see next, almost disjoint families are tightly related to Fréchet ideals.

Theorem 9.5 ([48]). *Let $\mathcal{I}$ be an ideal on X. The following statements are equivalent.*

(i) $\mathcal{I}$ is Fréchet.

(ii) There is an almost disjoint family $\mathcal{A}$ of infinite subsets of X such that $\mathcal{I} = \mathcal{A}^\perp$.

(iii) There is a family $\mathcal{A}$ of infinite subsets of X such that $\mathcal{I} = \mathcal{A}^\perp$.

Let us see some more examples of Fréchet ideals.

Example 9.6. *Consider the ideal $\mathcal{I}_{wf}$ generated by the well founded trees on $\mathbb{N}$ (see section 4). The orthogonal of $\mathcal{I}_{wf}$ is the ideal $\mathcal{I}_{do}$ generated by the finitely branching trees on $\mathbb{N}$, or equivalently, $\mathcal{I}_{do}$ consists of sets which are dominated by a branch:*

$$A \in \mathcal{I}_{do} \Leftrightarrow \exists \alpha \in \mathbb{N}^{\mathbb{N}} \forall s \in A \forall i < |s| (s(i) \leq \alpha(i)).$$

The ideal $\mathcal{I}_{wf}$ is a complete co-analytic Fréchet ideal, while the ideal $\mathcal{I}_{do}$ is easily seen to be $F_{\sigma\delta}$ (see [10, Example 2]*).*

9.1. Selective ideals

An ideal is *selective* if it is p$^+$and q$^+$. This is not the original definition given by Mathias [41] (who called them happy families) but it is a reformulation probably due to Kunen. The original first example of a selective ideal is the following:

Example 9.7 (Mathias [41]). *Let $\mathcal{A}$ be an analytic almost disjoint family of infinite subsets of $\mathbb{N}$. Then $\mathcal{I}(\mathcal{A})$ is a selective ideal.*

Next examples were found by Todorčević [55] in the realm of Banach spaces.

Example 9.8 ([57, Corollary 7.52]). *Let $f, f_n : X \to \mathbb{R}$ be pointwise bounded continuous functions, and suppose that $\{f_n : n \in \mathbb{N}\}$ accumulates to f. Let $\mathcal{I}_f$ be the ideal defined in Theorem 4.1, that is to say,*

$$E \in \mathcal{I}_f \text{ if, and only if, } f \notin \overline{\{f_n : n \in E\}}.$$

Then $\mathcal{I}_f$ is selective.

One of the reasons for being interested on selective ideals is due to the following.

Theorem 9.9 (Mathias [41]). *Every selective ideal is Ramsey.*

Selectivity is the combinatorial counterpart of the topological notion of bisequentiality (see [57, Theorem 7.53]) We only mention the following corollary of this fact which probably is due to Mathias [41].

Theorem 9.10. *Every selective analytic ideal is Fréchet.*

As we already said, if $\mathcal{I}$ is analytic, then $\mathcal{I}^\perp$ is co-analytic. Motivated by the study of Rosenthal compacta Krawczyk [35] and Todorčević [56], [57] have shown the following (see also [10]):

Theorem 9.11. *If $\mathcal{I}$ is a selective analytic ideal not countably generated, then $\mathcal{I}^\perp$ is a complete co-analytic set.*

The following examples illustrate the previous result.

Example 9.12. *Let $\mathcal{A}$ be the almost disjoint family given in Example 9.4(ii), and let $\mathcal{I}$ be $\mathcal{I}(\mathcal{A})$. Then $\mathcal{I}$ is selective (see Example 9.7) and it is analytic (actually it is F_σ), but it is not countable generated. Hence, $\mathcal{I}^\perp$ is $\mathbf{\Pi}^1_1$-complete (see* [10, Example 1]*).*

Example 9.13. *In Example 9.6 we presented the ideal $\mathcal{I}_{wf}$ generated by the well founded trees on $\mathbb{N}$ (see section 4). The orthogonal of $\mathcal{I}_{wf}$ is the ideal $\mathcal{I}_{do}$ consists of sets which are dominated by a branch. The ideal $\mathcal{I}_{wf}$ is a complete co-analytic set, while the ideal $\mathcal{I}_{do}$ is easily seen to be $F_{\sigma\delta}$, it is not countably generated and it is not selective (see* [10, Example 2]*).*

9.2. Orthogonal Borel families

Two families $\mathcal{A}$ and $\mathcal{B}$ of infinite subsets of $\mathbb{N}$ are called *orthogonal*, if $A \cap B$ is finite for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$ [54]. In this section we are interested in pairs of orthogonal families which are both Borel. An example is $\mathcal{A} = \emptyset \times \mathsf{Fin}$ and $\mathcal{B} = \mathsf{Fin} \times \emptyset$. The next theorem says this is the only possible such pair $(\mathcal{I}, \mathcal{I}^\perp)$ when one of them is a P-ideal.

Theorem 9.14 (Todorčević, [54, Theorem 7]). *Let $\mathcal{I}$ be an analytic P-ideal. Then $\mathcal{I}^\perp$ is countably generated if, and only if, $\mathcal{I}^\perp$ is Borel.*

In [27] was constructed a family $\mathfrak{B}$ of $\aleph_1$ non isomorphic Fréchet ideals such that both $\mathcal{I}$ and $\mathcal{I}^\perp$ are Borel. In fact, every ideal in $\mathfrak{B}$ is $F_{\sigma\delta}$. Let us recall its definition.

Let $\{K_n : n \in \mathbb{N}\}$ be a partition of X. For $n \in \mathbb{N}$, let $\mathcal{I}_n$ be an ideal on K_n. The direct sum, denoted by $\bigoplus_{n\in\mathbb{N}} \mathcal{I}_n$, is defined by

$$A \in \bigoplus_{n\in\mathbb{N}} \mathcal{I}_n \Leftrightarrow (\forall n \in \mathbb{N})(A \cap K_n \in \mathcal{I}_n).$$

For instance, if each $\mathcal{I}_n$ is isomorphic to Fin, then $\oplus_n \mathcal{I}_n$ is isomorphic to $\{\emptyset\} \times \mathsf{Fin}$. If each $\mathcal{I}_n$ is Fréchet, then $\bigoplus_n \mathcal{I}_n$ is also Fréchet.

The family $\mathfrak{B}$ is the smallest collection of ideals on $\mathbb{N}$ containing Fin and closed under countable direct sums and the operation of taking orthogonal. The family $\mathfrak{B}$ has some interesting properties.

Theorem 9.15 (Guevara-Uzcátegui [27]). *Let $\mathcal{I}$ be an analytic selective ideal on $\mathbb{N}$ and $A \subseteq \mathbb{N}$. The following statements are equivalent:*

(i) $\mathcal{I} \upharpoonright A$ *is countably generated.*

(ii) $\mathcal{I}^\perp \upharpoonright A \in \mathfrak{B}$.

(iii) $\mathcal{I}^\perp \upharpoonright A$ *is Borel.*

(iv) $\mathcal{I}_{wf} \not\to \mathcal{I}^\perp \upharpoonright A$.

Theorem 9.16 (Guevara-Uzcátegui [27]). *For every $A \subseteq \mathbb{N}^{<\omega}$, the following statements are equivalent:*

(i) $\mathcal{I}_{wf} \upharpoonright A$ *belongs to* $\mathfrak{B}$.

(ii) $\mathcal{I}_{wf} \upharpoonright A$ *is Borel.*

(iii) $\mathcal{I}_{wf} \not\to \mathcal{I}_{wf} \upharpoonright A$.

Another interesting co-analytic ideal is $WO(\mathbb{Q})$, the collection of well founded subsets of $WO(\mathbb{Q})$. For simplicity, we will write WO instead of $WO(\mathbb{Q})$. We first observe that $WO^\perp$ is the ideal of well founded subsets of $(\mathbb{Q}, <^*)$ where $<^*$ is the reversed order of $\mathbb{Q}$. In fact, the map $x \mapsto -x$ from $\mathbb{Q}$ onto $\mathbb{Q}$ is an isomorphism between WO and $WO^\perp$. In particular, WO is a Fréchet ideal. A linear order $(L, <)$ is said to be *scattered*, if it does not contain a order-isomorphic copy of $\mathbb{Q}$.

Theorem 9.17 (Guevara-Uzcátegui [27]). *For every $A \subseteq \mathbb{Q}$, the following statements are equivalent:*

(i) A *is scattered (with the order inherited from* $\mathbb{Q}$*).*

(ii) $WO \upharpoonright A$ *belongs to* $\mathfrak{B}$.

(iii) $WO \upharpoonright A$ *is Borel.*

(iv) $WO \not\to WO \upharpoonright A$.

It is known that every F_σ tall ideal is not Ramsey, and also that there is a co-analytic tall Ramsey ideal [31]. We have already stated the basic question of whether there is a Ramsey tall Borel ideal (see Question 8.8). A seemingly weaker question is

Question 9.18. Is there a non Fréchet Ramsey Borel (or analytic) ideal?

The only Borel Fréchet pairs $(\mathcal{I}, \mathcal{I}^{\perp})$ we are aware of are given by the ideals in $\mathfrak{B}$. So the natural question is:

Question 9.19. Is there a Borel Fréchet ideal with Borel orthogonal not isomorphic to an ideal in $\mathfrak{B}$?

A related question is the following

Question 9.20. Are there others $\mathbf{\Pi}^1_1$-complete Fréchet ideals satisfying the conclusion of theorem 9.16?

Since every Fréchet ideal is Katětov equivalent to Fin, then Katětov order is trivial among Fréchet ideals. But the Rudin-Keisler order is not trivial on Fréchet ideals [22], [21]. We say $\mathcal{I} \leq_{RK} \mathcal{J}$ if there is a function $f : \mathbb{N} \to \mathbb{N}$ such that $f^{-1}[E] \in \mathcal{J}$ if, and only if, $E \in \mathcal{I}$.

Theorem 9.21 (García-Ortiz [21])**.**

(i) There are strictly increasing $\leq_{RK}$-chains of Fréchet idelas of size $\mathfrak{c}^+$. Such chains can be constructed $\leq_{RK}$-above every Fréchet ideal.

(ii) For every infinite cardinal $\kappa < \mathfrak{c}$, there is a $\leq_{RK}$-antichain of size κ.

It is natural then to ask:

Question 9.22. How are the ideals in $\mathfrak{B}$ ordered according to $\leq_{RK}$?

F. Guevara [26] has classified the ideals in $\mathfrak{B}$ according to the Tukey order: Except for the countable generated, every ideal in $\mathfrak{B}$ is Tukey equivalent to $\mathbb{N}^{\mathbb{N}}$.

Obviously a Fréchet ideal cannot be topologically representable as it is not tall (see Theorem 6.6). It is easy to check that any Fréchet ideal is weakly selective. Thus the following question is appropriate.

Question 9.23. When is a Fréchet ideal countably separated?

F. Guevara [26] has shown that all ideals in $\mathfrak{B}$ are countably separated.

10. Uniform selection properties

As we have seen, most of the combinatorial properties for ideals are in fact selection properties. In this section we analyze the issue of whether the selector can be found Borel measurable. This question can be regarded as one instance of the classical uniformization problem in descriptive set theory: Let $B \subseteq X \times Y$ be a Borel set where X and Y are

Polish spaces. A Borel uniformization for B is a Borel function $F : X \to Y$ such that $(x, F(x)) \in B$ for all $x \in proy_X(B)$. It is well known that, in general, such Borel function does not exist (see section 18 of [33]).

As an illustration of the problem we are interested, let us consider the notion of tallness. Let $\mathcal{C}$ be a tall Borel (analytic, co-analytic) family of infinite subsets of $\mathbb{N}$. A very natural question is whether there is a Borel function $F : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ such that for all $A \subseteq \mathbb{N}$ infinite, $F(A)$ is an infinite subset of A and $F(A) \in \mathcal{C}$. That is to say, F witness in a Borel way that $\mathcal{C}$ is tall. In this case we can say that $\mathcal{C}$ is *uniformly tall* or that $\mathcal{C}$ has a *Borel selector*. This problem was studied in [24] and, in particular, they showed that there is a tall F_σ ideal which is not uniformly tall.

10.1. Uniform Ramsey properties

The main question we deal with in this section is whether it is possible to find in a Borel way an homogeneous subset of a given infinite sets. This could be briefly stated as whether Ramsey theorem holds uniformly. In the next section we shall see how it can be used to show that a given family is uniformly tall. Since selective ideals are Ramsey, we start discussing the uniform versions of the p$^+$and q$^+$ properties.

We say that a Borel ideal $\mathcal{I}$ is *uniformly* p^+ if there is a Borel function F from $(2^{\mathbb{N}})^{\mathbb{N}}$ into $2^{\mathbb{N}}$ such that whenever $(A_n)_n$ is a decreasing sequence of sets in $\mathcal{I}^+$, then $A = F((A_n)_n)$ is in $\mathcal{I}^+$ and $A \subseteq^* A_n$ for all $n \in \mathbb{N}$. We say that $\mathcal{I}$ is *uniformly* q^+, if there is a Borel function F from $(2^{\mathbb{N}})^{\mathbb{N}}$ into $2^{\mathbb{N}}$ such that whenever $\{K_n\}_n$ is a partition of a set A in $\mathcal{I}^+$ into finite sets, then $S = F((K_n)_n) \subseteq A$, S belongs to $\mathcal{I}^+$ and $|S \cap K_n| \leq 1$ for all n. If $\mathcal{I}$ is uniformly p$^+$and q$^+$, we say that $\mathcal{I}$ is *uniformly selective*.

The following is a uniform version of Theorem 5.4 and Example 9.7.

Theorem 10.1 ([24]). *Let $\mathcal{I}$ be an F_σ ideal. Then,*

(i) $\mathcal{I}$ is uniformly p^+.

(ii) If $\mathcal{I}$ is q^+, then it is uniformly q^+.

(iii) If $\mathcal{A}$ is an almost disjoint family of infinite subsets of $\mathbb{N}$ which is closed in $2^{\mathbb{N}}$, then $\mathcal{I}(\mathcal{A})$ is uniformly selective.

(iv) Fin *is uniformly selective.*

The previous result naturally suggests the following.

Question 10.2 ([24]). Is $\mathcal{I}(\mathcal{A})$ uniformly selective for any almost disjoint Borel family $\mathcal{A}$? More generally: is any Borel selective ideal uniformly selective?

Now we present some generalization of the Ramsey's theorem. We need some notation. For $s \in \mathsf{Fin}$ and $P \subseteq \mathbb{N}$ (finite or infinite), we write $s \sqsubseteq P$ when there is $n \in \mathbb{N}$ such that $s = P \cap \{0, 1, \cdots, n\}$, and we say that s is an initial segment of P.

Theorem 10.3 (Galvin's lemma). *Let $\mathcal{F} \subseteq \mathsf{Fin}$ and $M \in \mathbb{N}^{[\infty]}$. There is $N \subseteq M$ infinite such that one of the following statements holds:*

(i) For all $P \subseteq N$ infinite, there is $s \in \mathcal{F}$ such that $s \sqsubseteq P$.

(ii) $N^{[<\infty]} \cap \mathcal{F} = \emptyset$.

Any set N satisfying either (i) or (ii) will be called *$\mathcal{F}$-homogeneous*, and the collection of $\mathcal{F}$-homogeneous sets is denoted by $hom(\mathcal{F})$. Notice that if $\mathcal{F} \subseteq \mathbb{N}^{[2]}$, then we have a usual coloring $c : \mathbb{N}^{[2]} \to \{0,1\}$ by letting $c(s) = 1$ if, and only if, $s \in \mathcal{F}$. Then, $hom(\mathcal{F}) = hom(c)$. Notice also that the previous theorem in particular says that $hom(\mathcal{F})$ is a tall family for any $\mathcal{F} \subseteq \mathsf{Fin}$.

A collection $\mathcal{B} \subseteq \mathsf{Fin}$ is a *front* if it satisfies the following conditions: (i) Every two elements of $\mathcal{B}$ are $\sqsubseteq$-incomparable. (ii) Every infinite subset N of $\mathbb{N}$ has an initial segment in $\mathcal{B}$. A typical front is $\mathbb{N}^{[n]}$ for any $n \in \mathbb{N}$.

It is easy to verify that $hom(\mathcal{F})$ is co-analytic subset of $\mathbb{N}^{[\infty]}$ for every $\mathcal{F} \subseteq \mathsf{Fin}$. When $\mathcal{F} \subseteq \mathcal{B}$ and $\mathcal{B}$ is a front, $hom(\mathcal{F})$ is closed in $\mathbb{N}^{[\infty]}$. We do not know if there is $\mathcal{F}$ such that $hom(\mathcal{F})$ is not Borel.

A key result about the families $hom(\mathcal{F})$ is that they are uniformly tall when $\mathcal{F} \subseteq \mathcal{B}$ for some front $\mathcal{B}$. More precisely:

Theorem 10.4 ([24, Theorem 3.8]). *Let $\mathcal{B}$ be a front. There is a Borel map $S : 2^{\mathcal{B}} \times \mathbb{N}^{[\infty]} \to \mathbb{N}^{[\infty]}$ such that $S(\mathcal{F}, A)$ is an $\mathcal{F}$-homogeneous subset of A, for all $A \in \mathbb{N}^{[\infty]}$ and all $\mathcal{F} \subseteq \mathcal{B}$.*

f If we use the front $\mathbb{N}^{[2]}$ we obtain that the classical Ramsey theorem holds uniformly. We say that an ideal $\mathcal{I}$ is *uniformly Ramsey* if there is a Borel map $S : 2^{\mathbb{N}^{[2]}} \times \mathbb{N}^{[\infty]} \to \mathbb{N}^{[\infty]}$ such that for all $A \in \mathcal{I}^+$ and all $c : \mathbb{N}^{[2]} \to \{0,1\}$, $S(c, A) \in \mathcal{I}^+$ and it is a c-homogeneous subset of A. The following result is expected.

Theorem 10.5 ([24, Theorem 3.6]). *Every uniformly selective Borel ideal is uniformly Ramsey.*

It is also natural to wonder about when FinBW, Mon, $\mathbf{ws}$, $\mathbf{wp}^+$, etc. hold uniformly; this is left to the interested reader.

10.2. Uniformly tall ideals

From Theorem 10.4, using the front $\mathbb{N}^{[2]}$, we obtain that $hom(c)$ is a uniformly tall collection, and thus $\mathcal{I}_{hom(c)}$ is a uniformly tall ideal for any coloring c of pairs of natural numbers. It should be clear that if a collection $\mathcal{C}$ contains $hom(c)$ for some coloring c, then $\mathcal{C}$ is also uniformly tall. In fact, most of the examples we know of uniformly tall families are of that type. This could be regarded as a method for showing that a given family is uniformly tall (see example 10.6 below).

In particular, the random graph ideal $\mathcal{R}$ (see section 3) is uniformly tall. Thus, from the universal property of the random graph, we have that $\mathcal{R} \leq_K \mathcal{I}$ iff there is a $\mathcal{F} \subseteq [\mathbb{N}]^2$ such that $hom(\mathcal{F}) \subseteq \mathcal{I}$. Therefore, if $\mathcal{R} \leq_K \mathcal{I}$, then $\mathcal{I}$ has a Borel selector. That is the case with all examples studied in [29], [31]. Even Solecki's ideal $\mathcal{S}$ has a Borel selector [23], even though it is not known whether it is Katětov above $\mathcal{R}$ (see Question 7.6).

Example 10.6 ([24]). *The families of sets listed below are all uniformly tall. This is proved by finding a coloring $c : \mathbb{N}^{[2]} \to \{0,1\}$ such that hom(c) is a subset of the given family. The coloring used is the Sierpiński's coloring: Let $X = \{x_n : n \in \mathbb{N}\}$ be a countable set and $\preceq$ a total order on X. Define $c : X^{[2]} \to \{0,1\}$ by $c(\{x_n, x_m\}) = 0$ if, and only if, $n < m$ and $x_n \prec x_m$. The c-homogeneous sets are the $\prec$-monotone sequences in X:*

(i) $\mathsf{nwd}(\tau)$, *where (X,τ) is a Hausdorff countable space without isolated points.*

(ii) Let X be a compact metric space and $(x_n)_n$ be a sequence in X. Consider

$$\mathcal{C}(x_n)_n = \{A \subseteq \mathbb{N} : \ (x_n)_{n\in A} \text{ is convergent}\}.$$

(iii) Let $WO(\mathbb{Q})$ be the collection of all well-ordered subsets of $\mathbb{Q}$ respect the usual order. Let $WO(\mathbb{Q})^*$ the collection of well ordered subsets of $(\mathbb{Q}, <^*)$, where $<^*$ is the reversed order of the usual order of $\mathbb{Q}$. Then, $\mathcal{C} = WO(\mathbb{Q}) \cup WO(\mathbb{Q})^*$ is a tall family. Notice that $\mathcal{C}$ is $\mathbf{\Pi}^1_1$-complete.

It is not true that Galvin's theorem 10.3 holds uniformly. In fact, there is $\mathcal{F} \subseteq \mathsf{Fin}$ such that $hom(\mathcal{F})$ is not uniformly tall (see [24, Theorem 4.21]). Moreover, there is an F_σ tall ideal which is not uniformly tall (see [24, Theorem 4.18]). Since the proof of this fact is not constructive, we naturally have the following:

Question 10.7 ([24]). Find a concrete example of an F_σ tall ideal without a Borel selector.

Tall F_σ ideals are not q^+(otherwise they would be selective and thus Fréchet, see Theorems 5.4 and 9.10). This suggests the following:

Question 10.8. Is there a weakly selective (or q^+) tall Borel ideal without a Borel selector?

Property q^+ might be relevant as the next result suggests.

Theorem 10.9. *Let $\mathcal{I}$ be an analytic P-ideal. The following assertions are equivalent:*

(i) $\mathcal{I}$ is tall.

(ii) $\mathcal{I}$ has a continuous selector.

(iii) $\mathcal{I}$ is not q^+ at $\mathbb{N}$.

Since the generalized summable ideals $\mathcal{I}^G_h$ (see section 5.2) are somewhat similar to P-ideals, the previous result naturally suggests the following.

Question 10.10. Let $\mathcal{I}^G_h$ be a generalized summable ideal. Suppose $\mathcal{I}^G_h$ is tall. Is it uniformly tall?

The following result characterizes tall ideals with continuous selectors.

Theorem 10.11 (J. Grebík and M. Hrušák [23, Proposition 25]). *Let $\mathcal{I}$ be a Borel tall ideal. Then $\mathcal{I}$ has a continuous selector if, and only if, for every family $\{X_n : n \in \mathbb{N}\}$ of infinite subsets of $\mathbb{N}$ there is an $A \in \mathcal{I}$ such that $A \cap X_n \neq \emptyset$ for all $n \in \mathbb{N}$.*

These are the only results concerning the complexity of the selector functions. So we naturally wonder if there is a bound in the Borel complexity of the selector for Borel tall ideals.

Ideals admitting a topological representation (as defined in section 6.2) are tall and countably separated. So we have the following question (a negative answer of it will solve Question 10.8, as countably separated ideals are weakly selective [37, Proposition 4.3]).

Question 10.12. Suppose $\mathcal{I}$ is a co-analytic ideal with a topological representation. Is $\mathcal{I}$ uniformly tall?

Another question we could ask is whether there is a "simple basis" for the collection of all tall families. More precisely we have the following question:

Question 10.13. Let $\mathcal{C}$ be a tall family of infinite subsets of $\mathbb{N}$. Suppose that $\mathcal{C}$ is analytic or co-analytic. Is there $\mathcal{F} \subseteq \mathsf{Fin}$ such that $hom(\mathcal{F}) \subseteq \mathcal{C}$?

The restriction on the complexity is necessary as there is a $\mathbf{\Pi}^1_2$ tall ideal $\mathcal{I}$ such that $hom(\mathcal{F}) \not\subseteq \mathcal{I}$ for all $\mathcal{F} \subseteq \mathsf{Fin}$. In particular, $\mathcal{I}$ does not contain any closed hereditary tall set (see [24, Theorem 4.24]).

Some test families for the previous question are the following:

Example 10.14.

(a) Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two tall hereditary families with Borel selector. It is easy to verify that $\mathcal{C}_1 \cap \mathcal{C}_2$ is also uniformly tall. Let $\mathcal{B}_1$ and $\mathcal{B}_2$ two fronts on $\mathbb{N}$, and $\mathcal{F}_i \subseteq \mathcal{B}_i$, for $i = 0, 1$; is there a front $\mathcal{B}_3$ and $\mathcal{F}_3 \subseteq \mathcal{B}_3$ such that $hom(\mathcal{F}_3) \subseteq hom(\mathcal{F}_1) \cap hom(\mathcal{F}_2)$? Or more generally, given $\mathcal{F}_i \subseteq \mathsf{Fin}$, for $i = 0, 1$, is there $\mathcal{F}_3 \subseteq \mathsf{Fin}$ such that $hom(\mathcal{F}_3) \subseteq hom(\mathcal{F}_1) \cap hom(\mathcal{F}_2)$?

(b) Let $\mathcal{A}$ be an almost disjoint analytic family of infinite subsets of $\mathbb{N}$. Let $\mathcal{C}(\mathcal{A})$ be $\mathcal{I}(\mathcal{A}) \cup (\mathcal{I}(\mathcal{A}))^{\perp}$. Then $\mathcal{C}(\mathcal{A})$ is a $\mathbf{\Pi}^1_1$ tall family. The question would be for which families $\mathcal{A}$ there is $\mathcal{F} \subseteq \mathsf{Fin}$ such that $\mathrm{hom}(\mathcal{F}) \subseteq \mathcal{C}(\mathcal{A})$.

(c) Consider the following generalization of Example 10.6 (ii). Let K be a sequentially compact space, and $(x_n)_n$ be a sequence on K. Let

$$\mathcal{C}(x_n)_n = \{A \in \mathbb{N}^{[\infty]} : (x_n)_{n \in A} \text{ is convergent}\}.$$

Then $\mathcal{C}(x_n)_n$ is tall.

A particular interesting example is for K a separable Rosenthal compacta. By Debs' theorem [7], [8] *(see also* [9]*), in every Rosenthal compacta, $\mathcal{C}(x_n)_n$ is uniformly tall. When K is not first countable $\mathcal{C}(x_n)_n$, is a complete co-analytic subset of $\mathbb{N}^{[\infty]}$. We do not know if there is $\mathcal{F} \subseteq \mathsf{Fin}$ such that $hom(\mathcal{F}) \subseteq \mathcal{C}(x_n)_n$.*

10.3. Uniformly Fréchet ideals

A Fréchet ideal $\mathcal{I}$ on a countable set X is *uniformly Fréchet* if there is a Borel function $f : 2^X \to 2^X$ such that for all $A \subseteq X$ with $A \notin \mathcal{I}$, $F(A) \subseteq A$, $F(A)$ is infinite and $F(A) \in \mathcal{I}^\perp$.

Example 10.15 (Guevara [25]). *All ideals in* $\mathfrak{B}$ *(see section 9.2) are uniformly Fréchet.*

In view of the previous result, we have the following variant of Question 9.19.

Question 10.16 (Guevara [25]). Suppose $\mathcal{I}$ is an ideal such that $\mathcal{I}$ and $\mathcal{I}^\perp$ are Borel and uniformly Fréchet. Does $\mathcal{I}$ belong to $\mathfrak{B}$?

The definition of a uniformly Fréchet ideal does not requiere that it has to be a Borel ideal; however, we do not have an example of a non Borel uniformly Fréchet ideal.

Example 10.17 (Guevara [25]). *The ideals* $\mathcal{I}_c$ *and* $\mathcal{I}_{do}$ *are both uniformly Fréchet Borel ideals and* $\mathcal{I}_c^\perp$ *and* $\mathcal{I}_{do}^\perp$ *are not uniformly Fréchet.*

The previous example is a consequence of the following general fact.

Theorem 10.18 (Guevara [25]). *Let* $\mathcal{I}$ *be a Fréchet Borel ideal. If* $\mathcal{I}^\perp$ *is uniformly Fréchet, then* $\mathcal{I}^\perp$ *is Borel.*

Since Ramsey's theorem holds uniformly (see Theorem 10.4), we immediately have the following

Theorem 10.19. *Every uniformly Fréchet ideal is uniformly Ramsey.*

We have seen that every selective analytic ideal is Fréchet (see Theorem 9.10) and also that every F_σ selective ideal is uniformly selective. Thus we naturally ask the following:

Question 10.20. Is every uniformly selective F_σ ideal uniformly Fréchet? Or more generally, is every uniformly selective Borel ideal uniformly Fréchet?

We have already mentioned in Example 10.14 that $\mathcal{I} \cup \mathcal{I}^\perp$ is a tall familly for any ideal $\mathcal{I}$. It is easy to check that if $\mathcal{I}$ is uniformly Fréchet, then $\mathcal{I} \cup \mathcal{I}^\perp$ is uniformly tall. Thus we have the folllowing.

Question 10.21. Let $\mathcal{I}$ be a Borel Fréchet ideal such that $\mathcal{I} \cup \mathcal{I}^\perp$ is uniformly tall. Is $\mathcal{I}$ uniformly Fréchet?

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References

[1] Barbarski P., Filipów R., Mrożek N. and Szuca P., "When does the Katětov order imply that one ideal extends the other?", *Colloq. Math.* 130 (2013), No. 1, 91–102.

[2] Borodulin-Nadzieja P., Farkas B. and Plebanek G., "Representations of ideals in Polish groups and in Banach spaces", *J. Symb. Log.* 80 (2015), No. 4, 1268–1289.

[3] Camargo J. and Uzcátegui C., "Selective separability on spaces with an analytic topology", *Topology Appl.* 248 (2018), 176–191.

[4] Camargo J. and Uzcátegui C., "Some topological and combinatorial properties preserved by inverse limits", *Math. Slovaca* 69 (2019), No. 1, 171–184.

[5] Cameron P.J., "The random graph", in *The mathematics of Paul Erdös, II*, volume 14 of *Algorithms Combin.*, Springer (1997), 333–351.

[6] Ciesielski K. and Jasinski J., "Topologies making a given ideal nowhere dense or meager", *Topology Appl.* 63 (1995), 277–298.

[7] Debs G., "Effective properties in compact sets of Borel functions", *Mathematica* 34 (1987), 64–68.

[8] Debs G., "Borel extractions of converging sequences in compact sets of Borel functions", *J. Math. Anal. Appl.* 350 (2009), No. 2, 731–744.

[9] Dodos P., "Codings of separable compact subsets of the first Baire class", *Ann. Pure Appl. Logic* 142 (2006), 425–441.

[10] Dodos P. and Kanellopoulos V., "On pairs of definable orthogonal families", *Illinois J. Math.* 52 (2008), No. 1, 181–201.

[11] Farah I., "Ideals induced by Tsirelson submeasures", *Fund. Math.* 159 (1999), No. 3, 243–258.

[12] Farah I., *Analytic Quotiens*, volume 148 of *Mem. Amer. Math. Soc.*, Providence, Rhode Island, 2000.

[13] Farah I., "How many boolean algebras $\mathcal{P}(\mathbb{N})/I$ are there?", *Illinois J. Math.* 46 (2003), 999–1033.

[14] Farah I., "Luzin gaps", *Trans. Amer. Math. Soc.* 356 (2004), No. 6, 2197–2239.

[15] Farah I. and Solecki S., "Two $F_{\sigma\delta}$ ideals", *Proc. Amer. Math. Soc.* 131 (2003), No. 6, 1971–1975.

[16] Farkas B., Khomskii Y. and Vidnyánszky Z., "Almost disjoint refinements and mixing reals", *Fund. Math.* 242 (2018), No. 1, 25–48.

[17] Filipów R., Mrożek N., Reclaw I. and Szuca P., "$\mathcal{I}$-selection principles for sequences of functions", *J. Math. Anal. Appl.* 396 (2012), No. 2, 680–688.

[18] Filipów R. and Szuca P., "Rearrangement of conditionally convergent series on a small set", *J. Math. Anal. Appl.* 362 (2010), No. 1, 64–71.

[19] Fillipów R., Mrozek N., Reclaw I. and Szuca P., "Ideal convergence of bounded sequences", *J. symbolic Logic* 72 (2007), No. 2, 501–512.

[20] Fillipów R., Mrozek N., Reclaw I. and Szuca P., "Ideal version of Ramsey's theorem", *Czechoslovak Math. J.* 61 (2011), No. 2, 289–308.

[21] García-Ferreira S. and Rivera-Gómez J.E., "Comparing Fréchet-Urysohn filters with two pre-orders", *Topology Appl.* 225 (2017), 90–102.

[22] García-Ferreira S. and Rivera-Gómez J.E., "Ordering Fréchet-Urysohn filters", *Topology Appl.* 163 (2014), 128 – 141.

[23] Grebík J. and Hrušák M., "No minimal tall Borel ideal in the Katětov order", *preprint*, 2018.

[24] Grebík J. and Uzcátegui C., "Bases and Borel selectors for tall families", *J. Symb. Log.*, to appear, 2018.

[25] Guevara F., "Espacios uniformemente Fréchet", Master's thesis, Universidad de Los Andes, Mérida, Venezuela, 2011, 70 p.

[26] Guevara F., Personal communication, January, 2019.

[27] Guevara F. and Uzcátegui C., "Fréchet Borel ideals with Borel orthogonal", *Colloq. Math.* 152 (2018), No. 1, 141–163.

[28] Hrušák M., "Combinatorics of filters and ideals", in *Set theory and its applications*, volume 533 of *Contemp. Math.*, Amer. Math. Soc. (2011), 29–69.

[29] Hrušák M., "Katětov order on Borel ideals", *Arch. Math. Logic* 56 (2017), No. 7-8, 831–847.

[30] Hrušák M. and Meza-Alcántara D., "Comparison game on Borel ideals", *Comment. Math. Univ. Carolin.* 52 (2011), No. 2, 191–204.

[31] Hrušák M., Meza-Alcántara D., Thümmel E. and Uzcátegui C., "Ramsey type properties of ideals", *Ann. Pure Appl. Logic* 168 (2017), No. 11, 2022–2049.

[32] Hrušák M. and Meza-Alcántara D., "Universal submeasures and ideals", *Questions Answers Gen. Topology* 31 (2013), No. 2, 65–69.

[33] Kechris A.S., *Classical Descriptive Set Theory*, Springer-Verlag, New York, 1995.

[34] Klinga P. and Nowik A. "Extendability to summable ideals", *Acta Math. Hungar.* 152 (2017), No. 1, 150–160.

[35] Krawczyk A., "On the Rosenthal compacta and analytic sets", *Proc. Amer. Math. Soc.* 115 (1992), No. 4, 1095–1100.

[36] Kwela A. and Sabok M., "Topological representations", *J. Math. Anal. Appl.* 422 (2015), No. 2, 1434–1446.

[37] Kwela A. and Zakrzewski P., "Combinatorics of ideals—selectivity versus density", *Comment. Math. Univ. Carolin.* 58 (2017), No. 2, 261–266.

[38] Kwela M. and Nowik A., "Ideals of nowhere dense sets in some topologies on positive integers", *Topology Appl.* 248 (2018), 149 – 163.

[39] Laflamme C., "Filter games and combinatorial properties of strategies", in Tomek (ed.) Bartoszynski, editor, *Annual Boise extravaganza in set theory (BEST) conference,1992/1994*, volume 192 of *Contemp. Math.*, pages 51–67. American Mathematical Society., 1996.

[40] Laflamme C. and Leary C., “Filter games on ω and the dual ideal”, *Fund. Math.* 173 (2002), No. 2, 159–173.

[41] Mathias A.R.D., “Happy families”, *Ann. Math. Log.* 12 (1977), No. 1, 59–111.

[42] Mazur K., “F_σ-ideals and $\omega_1\omega_1^*$-gaps”, *Fund. Math.* 138 (1991), No. 2, 103–111.

[43] Meza-Alcántara D., “Ideals and filters on countable sets”, Thesis (Ph.D.), Universidad Nacional Autónoma de México, México, 2009, 139 p.

[44] Sabok M., Zapletal J, “Forcing properties of ideals of closed sets”, *J. Symb. Log.* 76 (2011), No. 3, 1075–1095.

[45] Sakai H., “On Katětov and Katětov-Blass orders on analytic P-ideals and Borel ideals”, *Arch. Math. Logic* 57 (2018), No. 3-4, 317–327.

[46] Samet N. and Tsaban B., “Superfilters, Ramsey theory, and van der Waerden’s Theorem”, *Topology Appl.*, 156 (2009), 2659–2669.

[47] Shibakov A., “On sequential analytic groups”, *Proc. Amer. Math. Soc.* 145 (2017), No. 9, 4087–4096.

[48] Simon P., “A hedgehog in a product”, *Acta. Univ. Carolin. Math. Phys.* 39 (1998), 147–153.

[49] Solecki S., “Analytic ideals and their applications”, *Ann. Pure Appl. Logic* 99 (1999), No. 1-3, 51–72.

[50] Solecki S., “Filters and sequences”, *Fund. Math.*, 163 (2000), 215–228.

[51] Solecki S., “Tukey reduction among analytic directed orders”, *Zb. Rad. (Beogr.)* 17 (2015), No. 25, (Selected topics in combinatorial analysis), 209–220.

[52] Solecki S. and Todorcevic S., “Avoiding families and Tukey functions on the nowhere-dense ideal”, *J. Inst. Math. Jussieu* 10 (2011), No. 2, 405–435.

[53] Solecki S. and Todorčević S., “Cofinal types of topological directed orders”, *Annales de l’institut Fourier*, 54 (2004), No. 6, 1877–1911.

[54] Todorčević S., “Analytic gaps”, *Fund. Math.* 150 (1996), No.1, 55–66.

[55] Todorčević S., *Topics in Topology*, Lecture Notes in Mathematics 1652, Springer, Berlin, 1997.

[56] Todorčević S., “Compacts sets of the first Baire class”, *Journal of the AMS* 12 (1999), No. 4, 1179–1212.

[57] Todorčević S., *Introduction to Ramsey spaces*, Annals of Mathematical Studies 174, Princeton University Press, 2010.

[58] Todorčević S. and Uzcátegui C., “Analytic topologies over countable sets”, *Topology Appl.* 111 (2001), No. 3, 299–326.

[59] Todorčević S. and Uzcátegui C., “Analytic k-spaces”, *Topology Appl.* 146-147 (2005), No. 1, 511–526.

[60] Todorčević S. and Uzcátegui C., “A nodec regular analytic space”, *Topology Appl.* 166 (2014), 85–91.

[61] Uzcátegui C., "On the complexity of the subspaces of S_ω", *Fund. Math.* 176 (2003), 1–16.

[62] Uzcátegui C. and Vielma J. "Alexandroff topologies viewed as closed sets in the Cantor cube", *Divulg. Mat.* 13 (2005), No. 1, 45–53.

[63] Veličković B., "A note on Tsirelson type ideals", *Fund. Math.* 159 (1999), No. 3, 259–268.

[64] Zafrany S., "On analytic filters and prefilters", *J. Symb. Log.* 55 (1990), No. 1, 315–322.