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On the property of Kelley for Hausdorff continua

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Abstract. We introduce the concepts *Hausdorff maximal limit continuum* and *Hausdorff strong maximal limit continuum*, for Hausdorff continua; these definitions extend the concepts of maximal limit continuum and strong maximal limit continuum, respectiveley, introduced by J. J. Charatonik and W. J. Charatonik in 1998 for metric continua [1, Definitions 2.2 and 2.3]. We show that in metric continua, being a maximal limit continuum is equivalent to being a Hausdorff maximal limit continuum. We also show that in metric continua, being a strong maximal limit continuum implies being a Hausdorff strong maximal limit continuum. Finally, we show an equivalence of having the property of Kelley, in terms of these new definitions, whose analog version for metric continua was given by J. J. Charatonik and W. J. Charatonik.
Keywords: Continuum, hyperspace, maximal limit continuum, property of Kelley, strong maximal limit continuum.
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La propiedad de Kelley para continuos de Hausdorff

Resumen. Introducimos los conceptos de *continuo límite maximal de Hausdorff* y *continuo límite maximal fuerte de Hausdorff*, para continuos de Hausdorff; estos conceptos extienden los ya definidos para continuos métricos: *continuo límite maximal* y *continuo límite maximal fuerte*, los cuales fueron dados por J. J. Charatonik y W. J. Charatonik en 1998 [1, Definitions 2.2 and 2.3]. Mostramos que en los continuos métricos el ser continuo límite maximal es equivalente a ser continuo límite maximal de Hausdorff. Probamos que en los continuos métricos todo continuo límite maximal fuerte es un continuo límite maximal fuerte de Hausdorff. Por último, mostramos una equivalencia para que un continuo de Huasdorff tenga la propiedad de Kelley en términos de estos nuevos conceptos, cuya versión análoga para continuos métricos fue dada por J. J. Charatonik y W. J. Charatonik.
Palabras clave: Continuo, continuo límite maximal, continuo límite maximal fuerte, hiperespacio, propiedad de Kelley.

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1. Introduction

A *continuum* is a compact, connected Hausdorff space with more than one point. A *metric continuum* is a continuum with a metric d that generates its topology.

The *property of Kelley* for metric continua was introduced by J. L. Kelley as *property 3.2* in [4, p. 26]; he used it to study the contractibility of hyperspaces (see [13, Chapter XVI] and [3, pp. 167–172]). In 1999, W. J. Charatonik [2, Definition 2.1] and W. Makuchowski [9, p. 124] extended the property of Kelley for continua; in particular, Charatonik shows an example of a homogeneous continuum that does not have the property of Kelley, and Makuchowski uses the property of Kelley to show that several definitions of local connectivity are equivalent in the hyperspace $C(X)$ of a continuum X having the property of Kelley. Concerning the generalization of some properties of metric continua to continua, S. Macías studied the property of Kelley for continua and introduces the uniform Effros property [6, p. 60]. In [7] and [8], the author proved that several properties of Jones' set function $\mathcal{T}$, valid for metric continua, hold for continua as well.

In 1998, J. J. Charatonik and W. J. Charatonik defined the concepts of *maximal limit continuum* and *strong maximal limit continuum*, for metric continua [1, Definitions 3.2 and 3.3]; the authors used those definitions to show several properties of continua having the property of Kelley, and to show that some properties are equivalent to the property of Kelley. In this paper we extend the metric concepts of maximal limit continuum and strong maximal limit continuum, to continua, which we call *Hausdorff maximal limit continuum* and *Hausdorff strong maximal limit continuum*, respectively. We show that in metric continua, the definition of maximal limit continuum is equivalent to the definition of Hausdorff maximal limit continuum (Theorem 4.8), and the definition of strong maximal limit continuum is stronger that the definition of Hausdorff strong maximal limit continuum (Proposition 4.13). To end the paper, we show that the equivalences of [1, Theorem 3.11] still hold under these new extensions (Theorem 4.19).

2. Preliminaries

Given a continuum X, we consider the collection of all nonempty closed subsets of X, which is denoted by 2^X; in other words,

$$2^X = \{A \subset X : A \text{ is a nonempty closed subset of } X\},$$

topologized with the *Vietoris topology*, which can be described as follows: for each $n \in \mathbb{N}$ and each finite collection $U_1, \ldots, U_n$ of open subsets of X, we define

$$\langle U_1, \ldots, U_n \rangle = \{A \in 2^X : A \subset \bigcup_{i=1}^{n} U_i \text{ and } A \cap U_i \neq \emptyset, \text{ for each } i \in \{1, \ldots, n\}\}.$$

The collection of all sets of the form $\langle U_1, \ldots, U_n \rangle$, is a basis for a topology for 2^X, which is called the Vietoris topology [11, Definition 1.7]. The set 2^X, endowed with the Vietoris topology, is called *hyperspace of closed subsets of* X. Also, we consider the collection of all subcontinua of X, denoted and defined by

$$C(X) = \{A \in 2^X : A \text{ is connected}\},$$

as a subspace of 2^X. The collection $C(X)$ is called the *hyperspace of subcontinua of* X. It is known that if X is a continuum, then 2^X and $C(X)$ are also continua [11, Theorems 4.9 and 4.10]. In this way, given a continuum X, we have that

$$2^{C(X)} = \{\mathcal{A} \subset C(X) : \mathcal{A} \text{ is a nonempty closed subset of } C(X)\}$$

and

$$C(C(X)) = \{\mathcal{A} \in 2^{C(X)} : \mathcal{A} \text{ is connected}\}$$

are also continua.

If X is a topological space, given a subset A of X, the interior of A is denoted by $int(A)$ and the closure of A by $cl(A)$. Given $A, B \in C(X)$ with $A \subset B$, we consider the collection

$$C(A, B) = \{K \in C(X) : A \subset K \subset B\}.$$

Remark 2.1. If A and B are subcontinua of X and $A \subset B$, then $C(A, B)$ is a nonempty connected closed subset of $C(X)$; in other words, $C(A, B)$ is an element of $C(C(X))$.

Proof. Let $L \in C(X) - C(A, B)$. We have that $L - B \neq \emptyset$ or $A - L \neq \emptyset$. If $L - B \neq \emptyset$, then there exists $x \in L - B$. Consider an open subset U of X such that $x \in U$ and $U \cap B = \emptyset$. Therefore, $L \in \langle U, X\rangle$ and $\langle U, X\rangle \cap C(A, B) = \emptyset$. Now, if $A - L \neq \emptyset$, then there exists $a \in A - L$. Consider an open subset V of X such that $a \in V$ and $cl(V) \cap L = \emptyset$. Notice that $L \in \langle X - cl(V)\rangle$. Let $L' \in \langle X - cl(V)\rangle$; then $a \notin L'$, consequently $A \nsubseteq L'$. It follows that $L' \notin C(A, B)$. Now, the proof that the hyperspace $C(X)$ is arcwise connected, by McWaters [10, Theorem, p. 1209] contains a proof that sets of the form $C(A, B)$ are connected. ☑

Let X be a continuum and $\mathcal{A} \subset 2^X$; we denote $\bigcup \mathcal{A} = \{x \in X :$ there exists $A \in \mathcal{A}$ such that $x \in A\}$.

Now, if X is a metric continuum, the hyperspace 2^X is also considered with the Hausdorff metric, which we denote by H. It is known that the Hausdorff metric of 2^X generates the Vietoris topology [13, (0.13) Theorem]. Given $r > 0$, $x \in X$ and $A \in 2^X$, let $B(r, x)$ be the open ball in X with center x and radius r and let $B_H(r, A)$ be the open ball in 2^X with center A and radius r; also, let $N_d(r, A) = \bigcup\{B(r, x) : x \in A\}$.

Given a function between spaces X and Y, $f : X \to Y$, $A \subset X$, and $B \subset Y$, let $f[A] = \{f(a) : a \in A\}$ the image of A under f, and $f^{-1}[B] = \{x \in X : f(x) \in B\}$, the inverse image of B under f.

3. *Limit superior of nets in* 2^X

A directed set is a pair $(D, \leq)$, where D is a nonempty set and $\leq$ is a partial order in D, such that for each $a, b \in D$, there exists $c \in D$ such that $a \leq c$ and $b \leq c$. A *net in* X is a function $f : D \to X$, where D is a directed set; we also denote a net f by $\{x_d\}_{d\in D}$, where $x_d = f(d)$, for each $d \in D$.

We recall that a net $\{x_d\}_{d\in D}$ in X *converges* to a point $x \in X$, if for each open subset U of X with $x \in U$, there exists $n \in D$ such that if $m \in D$ and $n \leq m$, then $x_m \in U$.

Given a continuum X, if $\{A_d\}_{d\in D}$ is a net in 2^X, we define the limit superior of $\{A_d\}_{d\in D}$ as follows:

$$\limsup\{A_d\}_{d\in D} = \{x \in X : \text{ for each open subset } U \text{ of } X \text{ with } x \in U \text{ and}$$
$$\text{for each } d \in D, \text{ there exists } m \in D \text{ such that } m \geq d \text{ and } U \cap A_m \neq \emptyset\}.$$

The limit superior of a net in 2^X was first considered by Mrówka [12, p. 237]. We present two properties of the limit superior that will be used in this paper.

Lemma 3.1 ([12, 4., p. 238]). *Let X be a continuum and let $\{A_d\}_{d\in D}$ be a net in 2^X. Then $\limsup\{A_d\}_{d\in D}$ is an element of 2^X.*

Lemma 3.2. *Let X be a continuum and let $\{A_d\}_{d\in D}$ be a net in $C(X)$ converging to $A \in C(X)$. Then $A \in \limsup\{C(A_d, X)\}_{d\in D}$.*

Proof. Let $\mathcal{U}$ be an open subset of $C(X)$ such that $A \in \mathcal{U}$. Since $\{A_d\}_{d\in D}$ converges to A, there exists $n \in D$ such that for each $m \in D$, if $n \leq m$, then $A_m \in \mathcal{U}$. Let $s \in D$ and choose $m \in D$ such that $s \leq m$ and $n \leq m$; then $A_m \in \mathcal{U}$ and $A_m \in C(A_m, X)$, hence $\mathcal{U} \cap C(A_m, X) \neq \emptyset$. We have proved that $A \in \limsup\{C(A_d, X)\}_{d\in D}$. ☑

4. Main results

In 1942, J. L. Kelley introduced the concept of *property of Kelley* for metric continua, originally called *property 3.2* [4, p. 26]:

Definition 4.1. Let X be a metric continuum with metric d. We say that X has the *property of Kelley* if for each $\varepsilon > 0$ there exists $\delta > 0$ such that for any $p, q \in X$, if $d(p, q) < \delta$, then for each $K \in C(\{p\}, X)$, there exists $L \in C(\{q\}, X)$ such that $H(K, L) < \varepsilon$.

Let us recall that H is the Hausdorff metric of the hyperspaces 2^X and $C(X)$; for $A \in 2^X$ and $r > 0$, $B_H(r, A)$ is the open ball in 2^X of radius r and center A.

In 1999, W. J. Charatonik [2, Definition 2.1] and W. Makuchowski [9, p. 124] introduced independently the concept of *property of Kelley* for continua, as follows:

Definition 4.2. Let X be a continuum and $p \in X$. We say that X has the *property of Kelley at* p, if for each $K \in C(\{p\}, X)$ and for each open subset $\mathcal{U}$ of $C(X)$ with $K \in \mathcal{U}$, there exists an open subset U of X with $p \in U$ such that if $q \in U$, then there exists $L \in C(\{q\}, X) \cap \mathcal{U}$. We say that X has the *property of Kelley* if it has the property of Kelley at each of its points.

In [14, p. 292] the author says that the property of Kelley and the pointwise version of property of Kelley are equivalent for metric continua. Since it is clear that the pointwise version of the property of Kelley on [14, p. 292] is equivalent to the pointwise version of the property of Kelley given in Definition 4.2, we have that Definition 4.2 is equivalent to Definition 4.1.

In the following theorem we summarize two characterizations of continua having the property of Kelley. The equivalence of *(1)* and *(3)* was proved by Wardle for metric continua in [14, (2.2) THEOREM], and Charatonik mentions without proof that the result is valid for continua [2, Proposition 2.3]; Makuchowski proved the equivalence of *(1)* and *(2)* [9, Theorem 11]. For the convenience of the reader, we prove *(2)* $\Rightarrow$ *(3)* and *(3)* $\Rightarrow$ *(1)* .

Theorem 4.3. *Let X be a continuum. Then the following conditions are equivalent:*

(1) X has the property of Kelley.

(2) For each open subset $\mathcal{U}$ of $C(X)$, it holds that $\bigcup\mathcal{U}$ is an open subset of X.

(3) The function $f : X \to 2^{C(X)}$, defined by $f(p) = C(\{p\}, X)$ for each $p \in X$, is continuous.

Proof. In this proof, since we consider the hyperspaces 2^X and $2^{C(X)}$, we will use the notation $\langle\cdot\rangle_{2^{C(X)}}$ to denote the basic open sets of the Vietoris topology in $2^{C(X)}$.

We prove that *(2)*$\Rightarrow$*(3)*. Let $\mathfrak{U}$ be an open subset of $2^{C(X)}$ and let $p \in f^{-1}[\mathfrak{U}]$. Notice that $C(\{p\}, X) = f(p) \in \mathfrak{U}$. Thus there are open subsets, $\mathcal{U}_1, \dots, \mathcal{U}_n$, of $C(X)$ such that $C(\{p\}, X) \in \langle\mathcal{U}_1, \dots, \mathcal{U}_n\rangle_{2^{C(X)}} \subset \mathfrak{U}$. By *(2)*, for each $i \in \{1, \dots, n\}$ we have that $\bigcup\mathcal{U}_i$ is an open subset of X. Since $C(\{p\}, X) \cap \mathcal{U}_i \neq \emptyset$, for each $i \in \{1, \dots, n\}$, there exists $A \in C(\{p\}, X) \cap \mathcal{U}_i$; since $p \in A \in \mathcal{U}_i$, we have that $p \in \bigcup\mathcal{U}_i$, for each $i \in \{1, \dots, n\}$.

Let $\mathcal{W} = \mathcal{U}_1 \cup \dots \cup \mathcal{U}_n$ and define $V' = \{v \in X : C(\{v\}, X) \subset \mathcal{W}\}$. We will show that V' is a neighborhood of p in X. In order to prove this assertion, first we prove that

$$C(X) = \mathcal{W} \cup \bigcup\{\langle U\rangle \cap C(X) : U \text{ is an open subset of } X, p \notin U \text{ and } p \in int(X - U)\}. \quad (1)$$

Notice that $C(\{p\}, X) \subset \mathcal{W}$. If $A \in C(X)$ and $A \notin C(\{p\}, X)$, then we choose two disjoint nonempty open subsets U and V of X such that $A \subset U$ and $p \in V$. Then $A \in \langle U\rangle \cap C(X)$ and $p \in V \subset int(X - U)$, and thus (1) follows.

On the other hand, by compactness of $C(X)$, there exist $k \in \mathbb{N}$ and $U_1, \dots, U_k$ open subsets of X such that $p \in int(X - U_i)$ for each $i \in \{1, \dots, k\}$ and

$$C(X) = \mathcal{W} \cup \bigcup_{i=1}^{k}(\langle U_i\rangle \cap C(X)).$$

Notice that $p \in int(X - \bigcup_{i=1}^{k} U_i)$. Let $v \in int(X - \bigcup_{i=1}^{k} U_i)$ and $B \in C(\{v\}, X)$; notice that if $B \in \langle U_i\rangle$, then $v \in U_i$; therefore $B \notin \bigcup_{i=1}^{k}(\langle U_i\rangle \cap C(X))$, and $B \in \mathcal{W}$. We have proved that $C(\{v\}, X) \subset \mathcal{W}$ for each $v \in int(X - \bigcup_{i=1}^{k} U_i)$, then V' is a neighborhood of p.

Now, let $V = V' \cap (\bigcap_{i=1}^{n}(\bigcup\mathcal{U}_i))$. We have that V is a neighborhood of p. We consider a point $v \in V$, and we will prove that $f(v) \in \langle\mathcal{U}_1, \dots, \mathcal{U}_n\rangle_{2^{C(X)}}$. Notice that $v \in V \subset V'$, therefore, $C(\{v\},X) \subset \mathcal{W} = \mathcal{U}_1 \cup \cdots \cup \mathcal{U}_n$. Let $i \in \{1,\ldots,n\}$; since $v \in V \subset \bigcap_{i=1}^{n}(\bigcup \mathcal{U}_i)$, then $v \in \bigcup \mathcal{U}_i$. Hence, there exists $D \in \mathcal{U}_i$ such that $v \in D \in \mathcal{U}_i$. This implies that $D \in C(\{v\},X) \cap \mathcal{U}_i \neq \emptyset$. Thus, $f(v) = C(\{v\},X) \in \langle \mathcal{U}_1,\ldots,\mathcal{U}_n \rangle_{2^{C(X)}} \subset \mathfrak{U}$, therefore $f(v) \in \mathfrak{U}$. We obtained that $f^{-1}[\mathfrak{U}]$ is an open subset of X, and this ends the implication *(2)⇒(3)*.

We see that *(3)⇒(1)*. Let $p \in X$, let $K \in C(\{p\},X)$ and let $\mathcal{U}$ be an open subset of $C(X)$ such that $K \in \mathcal{U}$. Notice that $C(\{p\},X) \in \langle \mathcal{U}, C(X) \rangle_{2^{C(X)}}$, because $C(\{p\},X) \subset \mathcal{U} \cup C(X) = C(X)$, $C(\{p\},X) \cap \mathcal{U} \neq \emptyset$ and $C(\{p\},X) \cap C(X) = C(\{p\},X) \neq \emptyset$. Let $U = f^{-1}[\langle \mathcal{U}, C(X) \rangle_{2^{C(X)}}]$; we have that U is an open subset of X and $p \in U$. Let $q \in U$, then $f(q) \in \langle \mathcal{U}, C(X) \rangle_{2^{C(X)}}$, hence $f(q) = C(\{q\},X)$ and $C(\{q\},X) \cap \mathcal{U} \neq \emptyset$. Let $L \in C(\{q\},X) \cap \mathcal{U}$, then $q \in L$ and $L \in \mathcal{U}$. Therefore, X has the property of Kelley. ☑

In 1998, J. J. Charatonik and W. J. Charatonik introduced the following definition for metric continua [1, Definition 3.2.]:

Definition 4.4. Let K be a subcontinuum of a metric continuum X. A subcontinuum $M \subset K$ is called a *maximal limit continuum in* K provided that there is a sequence $\{M_n\}_{n\in\mathbb{N}}$ of subcontinua of X, converging to M, such that for each sequence $\{M'_n\}_{n\in\mathbb{N}}$ of subcontinua of X with $M_n \subset M'_n$ for each $n \in \mathbb{N}$, if $\{M'_n\}_{n\in\mathbb{N}}$ converges to some $M' \in C(K)$, then $M' = M$.

For $\mathcal{U} \subset C(X)$, we define the collection

$$F(\mathcal{U}) = \{B \in C(X) : C(B,X) \cap \mathcal{U} \neq \emptyset\}.$$

We extend the concept *maximal limit continuum* for continua as follows:

Definition 4.5. Let K be a subcontinuum of a continuum X. A subcontinuum $M \subset K$ is called *Hausdorff maximal limit continuum in* K provided that for each subcontinuum $L \subset X$ with $M \subsetneq L \subset K$ there is an open subset $\mathcal{U}$ of $C(X)$ such that $L \in \mathcal{U}$ and the collection $F(\mathcal{U})$ is not neighborhood of M.

Remark 4.6. Let K be a subcontinuum of a continuum X. If $M \subset K$ is a Hausdorff maximal limit continuum in K, then for each subcontinuum $L \subset X$ with $M \subsetneq L \subset K$ there exists an open subset $\mathcal{U}$ of $C(X)$ such that $L \in \mathcal{U}$, and for each neighborhood $\mathcal{V} \subset C(X)$ of M there exists $B \in \mathcal{V}$ such that $C(B,X) \cap \mathcal{U} = \emptyset$.

Proof. Let L be a subcontinuum of X such that $M \subsetneq L \subset K$. By hypothesis, there is an open subset $\mathcal{U}$ of $C(X)$ such that $L \in \mathcal{U}$ and the collection $F(\mathcal{U})$ is not a neighborhood of M. Now, let $\mathcal{V} \subset C(X)$ be a neighborhood of M. Since $F(\mathcal{U})$ is not neighborhood of M, we obtain that $\mathcal{V} \nsubseteq F(\mathcal{U})$. Hence, there exists $B \in \mathcal{V}$ such that $C(B,X) \cap \mathcal{U} = \emptyset$. ☑

The following lemma follows from the definition.

Lemma 4.7. *Let K be a subcontinuum of a continuum X. Then K is a Hausdorff maximal limit continuum in K.*

We are going to prove that in metric continua Definitions 4.4 and 4.5 are equivalent:

Theorem 4.8. *Let X be a metric continuum, $K \in C(X)$ and $M \in C(K)$. Then the following statements are equivalent:*

(1) M is a maximal limit continuum in K.

(2) M is a Hausdorff maximal limit continuum in K.

Proof. We prove that *(1)⇒(2)*. Assume that X is a metric continuum that satisfies *(1)* but does not satisfy *(2)*. Let $\{M_n\}_{n\in\mathbb{N}}$ be the sequence converging to M, given by *(1)*. Since X does not satisfy (2), there exists $L \in C(X)$ such that $M \subsetneq L \subset K$ and for each $k \in \mathbb{N}$, we have that the set $F(B_H(\frac{1}{k}, L))$ is a neighborhood of M. Since $\{M_n\}_{n\in\mathbb{N}}$ converges to M, for each $k \in \mathbb{N}$ there exists $n_k \in \mathbb{N}$ such that for every $n \geq n_k$, we have that $M_n \in F(B_H(\frac{1}{k}, L))$. We can assume that the sequence $\{n_k\}_{k\in\mathbb{N}}$ is strictly increasing. Let $n \in \mathbb{N}$ with $n \geq n_1$, let $k \in \mathbb{N}$ such that $n_k \leq n < n_{k+1}$; thus we obtain that $M_n \in F(B_H(\frac{1}{k}, L))$, therefore $C(M_n, X) \cap B_H(\frac{1}{k}, L) \neq \emptyset$; choose $M'_n \in C(M_n, X) \cap B_H(\frac{1}{k}, L)$. Then the sequence $\{M'_n\}_{n=n_1}^{\infty}$ converges to $L \subset K$ and $L \neq M$, which is a contradiction.

We prove that *(2)⇒(1)*. For each $L \in C(M, K) - \{M\}$, let $\mathcal{U}_L$ be an open subset of $C(X)$ such that $L \in \mathcal{U}_L$ and $F(\mathcal{U}_L)$ is not neighborhood of M. Since $C(M, K)$ is a compact metric space, let $\{\mathcal{U}_i : i \in \mathbb{N}\}$ be a countable subcover of the cover $\{\mathcal{U}_L : L \in C(M, K) - \{M\}\}$ of the space $C(M, K) - \{M\}$. We can choose a sequence $\{M_n\}_{n\in\mathbb{N}}$ in $C(X)$ such that $H(M, M_n) \leq \frac{1}{n}$, for each $n \in \mathbb{N}$, and satisfying the following properties: $M_1 \notin F(\mathcal{U}_1)$, $M_2 \notin F(\mathcal{U}_2)$, $M_3 \notin F(\mathcal{U}_1)$, $M_4 \notin F(\mathcal{U}_2)$, $M_5 \notin F(\mathcal{U}_3)$, $M_6 \notin F(\mathcal{U}_1)$, $M_7 \notin F(\mathcal{U}_2)$, $M_8 \notin F(\mathcal{U}_3)$, $M_9 \notin F(\mathcal{U}_4)$, and so on inductively. We consider a sequence $\{M'_n\}_{n\in\mathbb{N}}$ in $C(X)$ convergent to some $M' \in C(K)$ and such that $M_n \subset M'_n$, for each $n \in \mathbb{N}$; hence, $M \subset M'$. Assume that $M \neq M'$, then there exists $k \in \mathbb{N}$ such that $M' \in \mathcal{U}_k$, and choose a subsequence $\{M'_{n_j}\}_{j\in\mathbb{N}}$ of the sequence $\{M'_n\}_{n\in\mathbb{N}}$ such that $M_{n_j} \notin F(\mathcal{U}_k)$, for each $j \in \mathbb{N}$. So, for each $j \in \mathbb{N}$, we have that $C(M_{n_j}, X) \cap \mathcal{U}_k = \emptyset$, then $M'_{n_j} \in C(X) - \mathcal{U}_k$. Since $C(X) - \mathcal{U}_k$ is a closed subset of $C(X)$, we have that $M' \in C(X) - \mathcal{U}_k$, which is a contradiction. We conclude that $M = M'$. ☑

In 1998, J. J. Charatonik and W. J. Charatonik introduced the following definition for metric continua [1, Definition 3.3.]:

Definition 4.9. Let K be a subcontinuum of a metric continuum X. A continuum $M \subset K$ is called a *strong maximal limit continuum in* K provided that there is a sequence $\{M_n\}_{n\in\mathbb{N}}$ of subcontinua of X converging to M, such that for each subsequence $\{M_{n_k}\}_{k\in\mathbb{N}}$ of $\{M_n\}_{n\in\mathbb{N}}$ and for each sequence $\{M'_k\}_{k\in\mathbb{N}}$ of subcontinua of X with $M_{n_k} \subset M'_k$ for each $k \in \mathbb{N}$, if $\{M'_k\}_{k\in\mathbb{N}}$ converges to some $M' \in C(K)$, then $M' = M$.

For $\mathcal{M}, \mathcal{U} \subset C(X)$, we define the collection

$$G(\mathcal{M}, \mathcal{U}) = \{B \in C(X) : C(B, X) \cap \mathcal{U} \cap (C(X) - \mathcal{M}) \neq \emptyset\}.$$

Concerning the definition of strong maximal limit continuum, we propose the following concept:

Definition 4.10. Let K be a subcontinuum of a continuum X. A subcontinuum $M \subset K$ is called *Hausdorff strong maximal limit continuum in* K provided that for each open subset $\mathcal{M}$ of $C(X)$ such that $M \in \mathcal{M}$, there exists an open subset $\mathcal{U}$ of $C(X)$ such that $C(M,K) \subset \mathcal{U}$ and the collection $G(\mathcal{M},\mathcal{U})$ is not neighborhood of M.

Remark 4.11. Let K be a subcontinuum of a continuum X. If $M \subset K$ is a Hausdorff strong maximal limit continuum in K, then for each open subset $\mathcal{M}$ of $C(X)$ such that $M \in \mathcal{M}$, there exists an open subset $\mathcal{U}$ of $C(X)$ such that $C(M,K) \subset \mathcal{U}$, and for each neighborhood $\mathcal{V}$ of M in $C(X)$ there exists $B \in \mathcal{V}$ such that $C(B,X) \cap \mathcal{U} \cap (C(X) - \mathcal{M}) = \emptyset$, equivalently $C(B,X) \cap \mathcal{U} \subset \mathcal{M}$.

Proof. Let $\mathcal{M}$ be an open subset of $C(X)$ such that $M \in \mathcal{M}$. By hypothesis, there is an open subset $\mathcal{U}$ of $C(X)$ such that $C(M,K) \subset \mathcal{U}$ and the collection $G(\mathcal{M},\mathcal{U})$ is not a neighborhood of M. Let $\mathcal{V}$ be a neighborhood of M in $C(X)$. Since $G(\mathcal{M},\mathcal{U})$ is not a neighborhood of M, we obtain that $\mathcal{V} \not\subseteq G(\mathcal{M},\mathcal{U})$. Hence, there exists $B \in \mathcal{V}$ such that $C(B,X) \cap \mathcal{U} \subset \mathcal{M}$. ☑

Lemma 4.12. *Let K be a subcontinuum of a continuum X. Then K is a Hausdorff strong maximal limit continuum in K.*

Proof. Let $\mathcal{M}$ be an open subset of $C(X)$ such that $K \in \mathcal{M}$. Since $C(K,K) = \{K\} \subset \mathcal{M}$, we consider $\mathcal{U} = \mathcal{M}$. Notice that $G(\mathcal{M},\mathcal{U}) = \{B \in C(X) : C(B,X) \cap \mathcal{U} \cap (C(X) - \mathcal{M}) \neq \emptyset\} = \emptyset$, which is not a neighborhood of K. ☑

We show that a strong maximal limit continuum is also a Hausdorff strong maximal limit continuum.

Proposition 4.13. *Let X be a metric continuum, $K \in C(X)$ and $M \in C(K)$. If M is a strong maximal limit continuum in K, then M is a Hausdorff strong maximal limit continuum in K.*

Proof. Let d be a metric on X, and let H be the Hausdorff metric on $C(X)$. Let $\{M_n\}_{n\in\mathbb{N}}$ be a sequence that witnesses that M is a strong maximal limit continuum in K. We consider an open subset $\mathcal{M}$ of $C(X)$ such that $M \in \mathcal{M}$. Suppose that for each open subset $\mathcal{U}$ of $C(X)$ such that $C(M,K) \subset \mathcal{U}$, we have that the collection $G(\mathcal{M},\mathcal{U})$ is a neighborhood of M. Now, for each $k \in \mathbb{N}$, let $\mathcal{U}_k = N_H(\frac{1}{k}, C(M,K))$. Notice that $C(M,K) \subset \mathcal{U}_k$, then the collection $G(\mathcal{M},\mathcal{U}_k)$ is a neighborhood of M. Inductively we can define a subsequence $\{M_{n_k}\}_{k\in\mathbb{N}}$ such that $M_{n_k} \in G(\mathcal{M},\mathcal{U}_k)$, for each $k \in \mathbb{N}$, hence $C(M_{n_k},X) \cap \mathcal{U}_k \cap (C(X) - \mathcal{M}) \neq \emptyset$. Consider an element $M'_k \in C(M_{n_k},X) \cap \mathcal{U}_k \cap (C(X) - \mathcal{M})$, for each $k \in \mathbb{N}$. Taking a subsequence if necessary, we may assume that $\{M'_k\}_{k\in\mathbb{N}}$ converges to an element $M' \in C(X)$. For each $k \in \mathbb{N}$, notice that $M'_k \in \mathcal{U}_k$ and $M'_k \in C(X) - \mathcal{M}$, then $M' \in C(M,K)$ and $M' \in C(X) - \mathcal{M}$, therefore $M' \neq M$, which contradicts the choice of the sequence $\{M_n\}_{n\in\mathbb{N}}$. ☑

Question 4.14. Let X be a metric continuum, $K \in C(X)$ and $M \in C(K)$, such that M is a Hausdorff strong maximal limit continuum in K. Is it true that M is a strong maximal limit continuum in K?

Proposition 4.15. *Let X be a continuum, let K be a subcontinuum of X, let $\{A_d\}_{d\in D}$ be a net in $C(X)$ converging to $A \in C(K)$ and let M be a maximal element in $C(K) \cap \limsup\{C(A_d, X)\}_{d\in D}$ (with respect to inclusion). Then M is a Hausdorff strong maximal limit continuum in K.*

Proof. Assume that M is not a Hausdorff strong maximal limit continuum in K. Then there exists an open subset $\mathcal{M}$ of $C(X)$ with $M \in \mathcal{M}$ such that for each open subset $\mathcal{U}$ of $C(X)$ with $C(M, K) \subset \mathcal{U}$, the collection $G(\mathcal{M},\mathcal{U})$ is a neighborhood of M. For each $L \in C(M, K) - \mathcal{M}$ we have that $M \subsetneq L$, and by maximality of M, we obtain that $L \notin \limsup\{C(A_d, X)\}_{d\in D}$. Thus there exists an open subset $\mathcal{V}_L$ of $C(X)$ such that $L \in \mathcal{V}_L$, and exists $d_L \in D$ such that if $m \geq d_L$, then $\mathcal{V}_L \cap C(A_m, X) = \emptyset$. By compactness of $C(M, K) - \mathcal{M}$ we obtain a finite subcover, $\{\mathcal{V}_1, \mathcal{V}_2, \ldots, \mathcal{V}_i\}$, for some $i \in \mathbb{N}$, of $\{\mathcal{V}_L : L \in C(M, K) - \mathcal{M}\}$ which covers $C(M, K) - \mathcal{M}$, and for each $\mathcal{V}_j$, there exists an element $d_j \in D$ such that if $m \geq d_j$, then $\mathcal{V}_{d_j} \cap C(A_m, X) = \emptyset$. We consider

$$\mathcal{U} = \mathcal{M} \cup \mathcal{V}_1 \cup \cdots \cup \mathcal{V}_i,$$

which is an open subset of $C(X)$ and $C(M, K) \subset \mathcal{U}$. Therefore, $G(\mathcal{M},\mathcal{U})$ is a neighborhood of M. Since $M \in \limsup\{C(A_d, X)\}_{d\in D}$, for each $d \in D$ there exists $m \in D$ such that $m \geq d$ and $G(\mathcal{M},\mathcal{U}) \cap C(A_m, X) \neq \emptyset$. Choose $n \in D$ such that $n \geq d_j$, for each $j \in \{1, 2, \ldots, i\}$, and $m \geq n$ such that $G(\mathcal{M},\mathcal{U}) \cap C(A_m, X) \neq \emptyset$. Consider $B \in G(\mathcal{M},\mathcal{U}) \cap C(A_m, X)$. Notice that $A_m \subset B$ and $C(B, X) \cap \mathcal{U} \cap (C(X) - \mathcal{M}) \neq \emptyset$. Since $C(B, X) \subset C(A_m, X)$, then $C(A_m, X) \cap \mathcal{U} \cap (C(X) - \mathcal{M}) \neq \emptyset$. Notice that $C(A_m, X) \cap \mathcal{V}_j = \emptyset$, for each $j \in \{1, 2, \ldots, i\}$, thus $C(A_m, X) \cap \mathcal{U} = C(A_m, X) \cap \mathcal{M}$, and $\emptyset \neq C(A_m, X) \cap \mathcal{U} \cap (C(X) - \mathcal{M}) = C(A_m, X) \cap \mathcal{M} \cap (C(X) - \mathcal{M}) = \emptyset$, which is a contradiction. This ends the proof that M is a Hausdorff strong maximal limit continuum in K. ☑

We will prove that Theorem 3.11 of [1] is still valid under these new definitions. We start with two lemmas.

Lemma 4.16. *Let X be a continuum with the property of Kelley, $K \in C(X)$ and $M \in C(K)$. If $M \neq K$, then M is not a Hausdorff maximal limit continuum in K.*

Proof. Suppose that a continuum X has the property of Kelley, $K \in C(X)$ and $M \in C(K)$ such that M is a Hausdorff maximal limit continuum in K and $M \neq K$. For $L = K$, by Remark 4.6 there exists an open subset $\mathcal{U}$ of $C(X)$ with $L \in \mathcal{U}$ such that for each neighborhood $\mathcal{V}$ of M in $C(X)$ there exists $B \in \mathcal{V}$ such that $C(B, X) \cap \mathcal{U} = \emptyset$. Consider $U_1, \ldots, U_n$ open subsets of X such that $L \in \langle U_1, \ldots, U_n\rangle \cap C(X) \subset \mathcal{U}$. Let

$$U = \bigcup(\langle U_1, \ldots, U_n\rangle \cap C(X)).$$

By Theorem 4.3, we have that U is an open subset of X; consequently $L \in \langle U\rangle \cap C(X)$, since $M \subset L$; then $M \subset U$. Let $\mathcal{V} = \langle U\rangle \cap C(X)$; then there exists $B \in \mathcal{V}$ such that $C(B,X)\cap\mathcal{U}=\emptyset$. Let $D\in\mathcal{V}$, choose a point $d\in D\subset U$, then there exists $J\in\langle U_1,\dots,U_n\rangle\cap C(X)$ such that $d\in J$. Since $D\cup J\subset U_1\cup\cdots\cup U_n$ and $(D\cup J)\cap U_i\neq\emptyset$, for each $i\in\{1,\dots,n\}$. It follows that $D\cup J\in\langle U_1,\dots,U_n\rangle\cap C(X)\subset\mathcal{U}$. Notice that $D\cup J\in C(D,X)$. Therefore, $C(D,X)\cap\mathcal{U}\neq\emptyset$, which is a contradiction. ☑

Lemma 4.17. *Let K be a subcontinuum of a continuum X and let M be a subcontinuum of K. If M is a Hausdorff strong maximal limit continuum in K, then M is a Hausdorff maximal limit continuum in K.*

Proof. Let $L\in C(M,K)$ with $L\neq M$. Consider disjoint nonempty open subsets $\mathcal{M}_1$ and $\mathcal{M}_2$ of $C(X)$ such that $M\in\mathcal{M}_1$ and $L\in\mathcal{M}_2$. By hypothesis, there exists an open subset $\mathcal{U}'$ of $C(X)$ such that $C(M,K)\subset\mathcal{U}'$ and the set $G(\mathcal{M}_1,\mathcal{U}')$ is not a neighborhood of M.

Let $\mathcal{U}=\mathcal{U}'\cap\mathcal{M}_2$. We will show that the set $F(\mathcal{U})$ is not a neighborhood of M, first we prove that:

$$F(\mathcal{U})\subset G(\mathcal{M}_1,\mathcal{U}'). \tag{2}$$

Indeed, let $B\in F(\mathcal{U})$; then $B\in C(X)$ and $C(B,X)\cap\mathcal{U}\neq\emptyset$, equivalently $C(B,X)\cap\mathcal{U}'\cap\mathcal{M}_2\neq\emptyset$, hence $C(B,X)\cap\mathcal{U}'\cap(C(X)-\mathcal{M}_1)\neq\emptyset$. We have proved that $B\in G(\mathcal{M}_1,\mathcal{U}')$. Thus (2) follows. Since $G(\mathcal{M}_1,\mathcal{U}')$ is not a neighborhood of M, then $F(\mathcal{U})$ is not a neighborhood of M. ☑

Corollary 4.18. *Let X be a continuum with the property of Kelley, $K\in C(X)$ and $M\in C(K)$. If $M\neq K$, then M is not a Hausdorff strong maximal limit continuum in K.*

To conclude this paper, we prove that Theorem 3.11 of [1] is still valid under these new definitions.

Theorem 4.19. *Let X be a continuum; then, the following statements are equivalent:*

(1) X has the property of Kelley.

(2) For each subcontinuum K of X, the only Hausdorff maximal limit continuum in K is K itself.

(3) For each subcontinuum K of X, the only Hausdorff strong maximal limit continuum in K is K itself.

Proof. By Lemma 4.7 and Lemma 4.16, we have that *(1)⇒(2)*.

By Lemma 4.12 and Lemma 4.17, we have that *(2)⇒(3)*.

We prove *(3)⇒(1)*. Consider $\mathcal{U}$ an open subset of $C(X)$. By Theorem 4.3, it is sufficient to show that $\bigcup\mathcal{U}$ is an open subset of X. Asumme that $\bigcup\mathcal{U}$ is not open, and choose a point $p\in\bigcup\mathcal{U}-int(\bigcup\mathcal{U})$. Let

$$\mathcal{D}=\{V\subset X : V \text{ is an open subset of } X \text{ and } p\in V\}$$

be a directed set, where $V_1 \leq V_2$ if $V_2 \subset V_1$, for each $V_1, V_2 \in \mathcal{D}$. For each $V \in \mathcal{D}$, choose $p_V \in V - \bigcup \mathcal{U}$. Notice that the net $\{p_V\}_{V \in \mathcal{D}}$ converges p in X and $p_V \notin \bigcup \mathcal{U}$, for each $V \in \mathcal{D}$. Let $K \in \mathcal{U}$ such that $p \in K$. Since the net $\{\{p_V\}\}_{V \in \mathcal{D}}$ converges to $\{p\}$ in $C(X)$, by Lemma 3.2 we have that $\{p\} \in \limsup\{C(\{p_V\}, X)\}_{V \in \mathcal{D}}$. Therefore $C(K) \cap \limsup\{C(\{p_V\}, X)\}_{V \in \mathcal{D}} \neq \emptyset$. By the Kuratowski-Zorn Lemma [5, p. 33], there exists $M \in C(K) \cap \limsup\{C(\{p_V\}, X)\}_{V \in \mathcal{D}}$ maximal with respect to inclusion. By Proposition 4.15, we have that M is a Hausdorff strong maximal limit continuum in K. By *(3)*, we have that $M = K$. It follows that $K \in \limsup\{C(\{p_V\}, X)\}_{V \in \mathcal{D}}$; since $\mathcal{U}$ is an open subset of $C(X)$ and $K \in \mathcal{U}$, then for each $R \in \mathcal{D}$, there exists $S \in \mathcal{D}$ such that $S \geq R$ and $\mathcal{U} \cap C(\{p_S\}, X) \neq \emptyset$. Choose $B \in \mathcal{U} \cap C(\{p_S\}, X)$, then $p_S \in B \in \mathcal{U}$. Hence $p_S \in \bigcup \mathcal{U}$, which is a contradiction. We have proved that $\bigcup \mathcal{U}$ is an open subset of X. This ends the proof of the theorem. ☑

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