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Generalized Fractional Ostrowski and Grüss type inequalities involving several Banach algebra valued function

George A. Anastassiou ✉,
Department of Mathematical Sciences, University of Memphis, Memphis, TN 38152, U.S.A.

Abstract. Using generalized Caputo fractional left and right vectorial Taylor formulae, we establish mixed fractional Ostrowski and Grüss type inequalities involving several Banach algebra valued functions. The estimates are with respect to all norms $\|\cdot\|_p$, $1 \leq p \leq \infty$.

Keywords: Generalized fractional derivative, generalized fractional inequalities, Ostrowski inequality, Grüss inequality, Banach algebra.

MSC2010: 26A33, 26D10, 26D15.

Desigualdades fraccionarias generalizadas de tipo Ostrowski y Grüss que involucran varias funciones valoradas del álgebra de Banach

Resumen. Usando fórmulas de Taylor vectoriales fraccionarias izquierda y derecha de Caputo generalizadas, establecemos desigualdades fraccionarias mixtas de tipo Ostrowski y Grüss que involucran varias funciones valoradas del álgebra de Banach. Las estimaciones son con respecto a todas las normas $\|\cdot\|_p$, $1 \leq p \leq \infty$.

Palabras clave: Derivada fraccionaria generalizada, desigualdades fraccionarias generalizadas, desigualdad de Ostrowski, desigualdad de Grüss, álgebra de Banach.

E-mail: ganastss@memphis.edu ✉.
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1. Introduction

The following results motivate our work.

Theorem 1.1 (1938, Ostrowski [10]). *Let $f : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b) whose derivative $f' : (a, b) \to \mathbb{R}$ is bounded on (a, b), i.e., $\|f'\|_\infty^{\sup} := \sup_{t\in(a,b)} |f'(t)| < +\infty$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(t)\, dt - f(x) \right| \leq \left[\frac{1}{4} + \frac{\left(x - \frac{a+b}{2}\right)^2}{(b-a)^2} \right] (b-a) \|f'\|_\infty^{\sup}, \tag{1}$$

for any $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible.

Ostrowski type inequalities have great applications to integral approximations in Numerical Analysis.

Theorem 1.2 (1882, Čebyšev [6]). *Let $f, g : [a, b] \to \mathbb{R}$ be absolutely continuous functions with $f', g' \in L_\infty([a, b])$. Then*

$$\left| \frac{1}{b-a} \int_a^b f(x) g(x)\, dx - \left(\frac{1}{b-a} \int_a^b f(x)\, dx \right) \left(\frac{1}{b-a} \int_a^b g(x)\, dx \right) \right|$$

$$\leq \frac{1}{12} (b-a)^2 \|f'\|_\infty \|g'\|_\infty. \tag{2}$$

The above integrals are assumed to exist.

The related Grüss type inequalities have many applications to Probability Theory. We presented also ([4], Ch. 8,9) mixed fractional Ostrowski and Grüss-Cebysev type inequalities for several functions, acting to all possible directions. The estimates involve the left and right Caputo fractional derivatives. See also the monographs written by the author [2], Chapters 24-26 and [3], Chapters 2-6.

In this article we generalize [4], Ch. 8,9 for several Banach algebra valued functions. Now our left and right Caputo fractional derivatives are for Banach space valued functions and our integrals are of Bochner type. Several applications finish this article. Inspiration came also from [7], [8].

2. Vectorial background fractional calculus

Here all come from [5]. We need:

Definition 2.1 ([5], p. 106). Let $\alpha > 0$, $\lceil \alpha \rceil = n$, $\lceil \cdot \rceil$ the ceiling of the number. Let $f \in C^n([a, b], X)$, where $[a, b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ is a Banach space. Also let $g \in C^1([a, b])$, strictly increasing such that $g^{-1} \in C^n([g(a), g(b)])$.

We define the left generalized g-fractional derivative X-valued of f of order α as follows:

$$\left(D_{a+;g}^{\alpha} f\right)(x) := \frac{1}{\Gamma(n-\alpha)} \int_a^x (g(x) - g(t))^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} (g(t))\, dt, \tag{3}$$

$\forall\, x \in [a,b]$, where Γ is the gamma function. The last integral is of Bochner type ([1], pp. 422-428; [9]).

If $\alpha \notin \mathbb{N}$, by Theorem 4.10 ([5], p. 98), we have that $\left(D^{\alpha}_{a+;g} f\right) \in C([a,b],X)$.

We set

$$D^{n}_{a+;g} f(x) := \left(\left(f \circ g^{-1}\right)^{n} \circ g\right)(x) \in C([a,b],X), \quad n \in \mathbb{N}, \tag{4}$$

$$D^{0}_{a+;g} f(x) = f(x), \quad \forall\, x \in [a,b].$$

When $g = id$, then

$$D^{\alpha}_{a+;g} f = D^{\alpha}_{a+;id} f = D^{\alpha}_{*a} f, \tag{5}$$

the usual left X-valued Caputo fractional derivative, see [5], Ch. 1.

We also need:

Definition 2.2 ([5], p. 107)**.** Let $\alpha > 0$, $\lceil \alpha \rceil = n$, $\lceil \cdot \rceil$ the ceiling of the number. Let $f \in C^{n}([a,b],X)$, where $[a,b] \subset \mathbb{R}$, and $(X, \|\cdot\|)$ a Banach space. Let $g \in C^{1}([a,b])$, strictly increasing such that $g^{-1} \in C^{n}([g(a), g(b)])$.

We define the right generalized g-fractional derivative X-valued of f of order α as follows:

$$\left(D^{\alpha}_{b-;g} f\right)(x) := \frac{(-1)^{n}}{\Gamma(n-\alpha)} \int_{x}^{b} (g(t) - g(x))^{n-\alpha-1} g'(t) \left(f \circ g^{-1}\right)^{(n)} (g(t))\, dt, \tag{6}$$

$\forall\, x \in [a,b]$. The last integral is of Bochner type.

If $\alpha \notin \mathbb{N}$, by Theorem 4.11 ([5], p. 101), we have that $\left(D^{\alpha}_{b-;g} f\right) \in C([a,b],X)$.

We set

$$D^{n}_{b-;g} f(x) := (-1)^{n} \left(\left(f \circ g^{-1}\right)^{n} \circ g\right)(x) \in C([a,b],X), \quad n \in \mathbb{N}, \tag{7}$$

$$D^{0}_{b-;g} f(x) := f(x), \quad \forall\, x \in [a,b].$$

When $g = id$, then

$$D^{\alpha}_{b-;g} f(x) = D^{\alpha}_{b-;id} f(x) = D^{\alpha}_{b-} f, \tag{8}$$

the usual right X-valued Caputo fractional derivative, see [5], Ch. 2.

We mention the following generalized fractional Taylor formulae with integral remainders over Banach spaces.

Theorem 2.3 ([5], p. 107)**.** *Let $\alpha > 0$, $n = \lceil \alpha \rceil$, and $f \in C^{n}([a,b],X)$, where $[a,b] \subset \mathbb{R}$ and $(X, \|\cdot\|)$ a Banach space. Let $g \in C^{1}([a,b])$, strictly increasing such that $g^{-1} \in C^{n}([g(a), g(b)])$, $a \le x \le b$. Then*

$$f(x) = f(a) + \sum_{i=1}^{n-1} \frac{(g(x) - g(a))^{i}}{i!} \left(f \circ g^{-1}\right)^{(i)} (g(a)) +$$

$$\frac{1}{\Gamma(\alpha)} \int_{a}^{x} (g(x) - g(t))^{\alpha-1} g'(t) \left(D^{\alpha}_{a+;g} f\right)(t)\, dt =$$

$$f(a)+\sum_{i=1}^{n-1}\frac{(g(x)-g(a))^{i}}{i!}\left(f\circ g^{-1}\right)^{(i)}(g(a))+ \tag{9}$$

$$\frac{1}{\Gamma(\alpha)}\int_{g(a)}^{g(x)}(g(x)-z)^{\alpha-1}\left(\left(D_{a+;g}^{\alpha}f\right)\circ g^{-1}\right)(z)\,dz.$$

We also mention:

Theorem 2.4 ([5], p. 108). *Let $\alpha>0$, $n=\lceil\alpha\rceil$, and $f\in C^{n}([a,b],X)$, where $[a,b]\subset\mathbb{R}$ and $(X,\|\cdot\|)$ a Banach space. Let $g\in C^{1}([a,b])$, strictly increasing such that $g^{-1}\in C^{n}([g(a),g(b)])$, $a\leq x\leq b$. Then*

$$f(x)=f(b)+\sum_{i=1}^{n-1}\frac{(g(x)-g(b))^{i}}{i!}\left(f\circ g^{-1}\right)^{(i)}(g(b))+$$

$$\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(g(t)-g(x))^{\alpha-1}g'(t)\left(D_{b-;g}^{\alpha}f\right)(t)\,dt=$$

$$f(b)+\sum_{i=1}^{n-1}\frac{(g(x)-g(b))^{i}}{i!}\left(f\circ g^{-1}\right)^{(i)}(g(b))+ \tag{10}$$

$$\frac{1}{\Gamma(\alpha)}\int_{g(x)}^{g(b)}(z-g(x))^{\alpha-1}\left(\left(D_{b-;g}^{\alpha}f\right)\circ g^{-1}\right)(z)\,dz.$$

If $0<\alpha\leq 1$, then the sums in (9), (10) disappear.

Also in (9), (10), we have that

$$\left\|\left\|\left(D_{a+;g}^{\alpha}f\right)\circ g^{-1}\right\|\right\|_{\infty,[g(a),g(b)]},\ \ \left\|\left\|\left(D_{b-;g}^{\alpha}f\right)\circ g^{-1}\right\|\right\|_{\infty,[g(a),g(b)]}<\infty.$$

3. Banach Algebras background

All here come from [11].

We need:

Definition 3.1 ([11], p. 245). A complex algebra is a vector space A over the complex filed $\mathbb{C}$ in which a multiplication is defined that satisfies

$$x(yz)=(xy)z, \tag{11}$$

$$(x+y)z=xz+yz,\ \ x(y+z)=xy+xz, \tag{12}$$

and

$$\alpha(xy)=(\alpha x)y=x(\alpha y), \tag{13}$$

for all x,y and z in A and for all scalars α.

Additionally if A is a Banach space with respect to a norm that satisfies the multiplicative inequality

$$\|xy\| \leq \|x\| \|y\| \quad (x \in A, \, y \in A) \tag{14}$$

and if A contains a unit element e such that

$$xe = ex = x \quad (x \in A) \tag{15}$$

and

$$\|e\| = 1, \tag{16}$$

then A is called a Banach algebra.

A is commutative iff $xy = yx$ for all $x, y \in A$.

Remark 3.2. Commutativity of A will be explicited stated when needed.

There exists at most one $e \in A$ that satisfies (15).

Inequality (14) makes multiplication to be continuous, more precisely left and right continuous, see [11], p. 246.

Multiplication in A is not necessarily the numerical multiplication, it is something more general and it is defined abstractly, that is for $x, y \in A$ we have $xy \in A$, e.g. composition or convolution, etc.

For nice examples about Banach algebras see [11], p. 247-248, $S(\Lambda)$ 10.3.

Remark 3.3. Next we mention about integration of A-valued functions, see [11], p. 259, $S(\Lambda)$ 10.22:

If A is a Banach algebra and f is a continuous A-valued function on some compact Hausdorff space Q on which a complex Borel measure μ is defined, then $\int f d\mu$ exists and has all the properties that were discussed in Chapter 3 of [11], simply because A is a Banach space. However, an additional property can be added to these, namely: If $x \in A$, then

$$x \int_Q f \, d\mu = \int_Q x f(p) \, d\mu(p) \tag{17}$$

and

$$\left(\int_Q f \, d\mu \right) x = \int_Q f(p) \, x \, d\mu(p) \, . \tag{18}$$

The Bochner integrals we will involve in our article follow (17) and (18). Also, let $f \in C([a, b], X)$, where $[a, b] \subset \mathbb{R}$, $(X, \|\cdot\|)$ is a Banach space. By [5], p. 3, f is Bochner integrable.

4. Main Results

We start with mixed generalized fractional Ostrowski type inequalities for several functions over a Banach algebra. A uniform estimate follows.

Theorem 4.1. *Let* $(A, \|\cdot\|)$ *be a Banach algebra,* $x_0 \in [a,b] \subset \mathbb{R}$, $\alpha > 0$, $n = \lceil \alpha \rceil$, $f_i \in C^n([a,b], A)$, $i = 1,...,r \in \mathbb{N} - \{1\}$; $g \in C^1([a,b])$, *strictly increasing such that* $g^{-1} \in C^n([g(a), g(b)])$, *with* $\left(f_i \circ g^{-1}\right)^{(k)}(g(x_0)) = 0$, $k = 1,...,n-1$; $i = 1,...,r$. *Denote by*

$$\theta(f_1,...,f_r)(x_0) :=$$

$$\sum_{i=1}^{r}\left[\int_a^b \left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) f_i(x)\,dx - \left(\int_a^b \left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) dx\right) f_i(x_0)\right]. \tag{19}$$

Then

$$\|\theta(f_1,...,f_r)(x_0)\| \leq \frac{1}{\Gamma(\alpha+1)} \sum_{i=1}^{r} \left[\left[\left\|\left\|\left(D^{\alpha}_{x_0-;g} f_i\right) \circ g^{-1}\right\|\right\|_{\infty,[g(a),g(x_0)]}\right.\right.$$

$$\left.(g(x_0) - g(a))^{\alpha} \left(\int_a^{x_0} \left(\prod_{\substack{j=1\\ j\neq i}}^{r} \|f_j(x)\|\right) dx\right)\right] + \tag{20}$$

$$\left.\left[\left\|\left\|\left(D^{\alpha}_{x_0+;g} f_i\right) \circ g^{-1}\right\|\right\|_{\infty,[g(x_0),g(b)]} (g(b) - g(x_0))^{\alpha} \left(\int_{x_0}^b \left(\prod_{\substack{j=1\\ j\neq i}}^{r} \|f_j(x)\|\right) dx\right)\right]\right].$$

Proof. Since $\left(f_i \circ g^{-1}\right)^{(k)}(g(x_0)) = 0$, $k = 1,...,n-1$; $i = 1,...,r$, we have by Theorem 2.3 that

$$f_i(x) - f_i(x_0) = \frac{1}{\Gamma(\alpha)} \int_{g(x_0)}^{g(x)} (g(x) - z)^{\alpha-1} \left(\left(D^{\alpha}_{x_0+;g} f_i\right) \circ g^{-1}\right)(z)\,dz, \tag{21}$$

$\forall\, x \in [x_0, b]$,

and by Theorem 2.4 that

$$f_i(x) - f_i(x_0) = \frac{1}{\Gamma(\alpha)} \int_{g(x)}^{g(x_0)} (z - g(x))^{\alpha-1} \left(\left(D^{\alpha}_{x_0-;g} f_i\right) \circ g^{-1}\right)(z)\,dz, \tag{22}$$

$\forall\, x \in [a, x_0]$;

for all $i = 1,...,r$.

Left multiplying (21) and (22) with $\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)$ we get, respectively,

$$\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) f_i(x) - \left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right) f_i(x_0) = \tag{23}$$

$$\frac{\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)}{\Gamma(\alpha)} \int_{g(x_0)}^{g(x)} (g(x)-z)^{\alpha-1} \left(\left(D_{x_0+;g}^{\alpha} f_i\right) \circ g^{-1}\right)(z)\, dz,$$

$\forall\ x \in [x_0, b]$,

and

$$\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x) - \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x_0) = \tag{24}$$

$$\frac{\left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right)}{\Gamma(\alpha)} \int_{g(x)}^{g(x_0)} (z-g(x))^{\alpha-1} \left(\left(D_{x_0-;g}^{\alpha} f_i\right) \circ g^{-1}\right)(z)\, dz,$$

$\forall\ x \in [a, x_0]$;

for all $i = 1, ..., r$.

Adding (23) and (24) as separate groups, we obtain

$$\sum_{i=1}^{r} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x) - \sum_{i=1}^{r} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x_0) =$$

$$\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{r} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) \int_{g(x_0)}^{g(x)} (g(x)-z)^{\alpha-1} \left(\left(D_{x_0+;g}^{\alpha} f_i\right) \circ g^{-1}\right)(z)\, dz, \tag{25}$$

$\forall\ x \in [x_0, b]$,

and

$$\sum_{i=1}^{r} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x) - \sum_{i=1}^{r} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x_0) =$$

$$\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{r} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) \int_{g(x)}^{g(x_0)} (z-g(x))^{\alpha-1} \left(\left(D_{x_0-;g}^{\alpha} f_i\right) \circ g^{-1}\right)(z)\, dz, \tag{26}$$

$\forall\ x \in [a, x_0]$.

Next we integrate (25) and (26) with respect to $x \in [a, b]$. We have

$$\sum_{i=1}^{r} \int_{x_0}^{b} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) f_i(x)\, dx - \sum_{i=1}^{r} \left(\int_{x_0}^{b} \left(\prod_{\substack{j=1\\j\neq i}}^{r} f_j(x)\right) dx\right) f_i(x_0) =$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{\alpha-1}\left(\left(D_{x_0+;g}^{\alpha}f_i\right)\circ g^{-1}\right)(z)\,dz\right)dx\right], \tag{27}$$

and

$$\sum_{i=1}^{r}\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)f_i(x)\,dx-\sum_{i=1}^{r}\left(\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)dx\right)f_i(x_0)=$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{\alpha-1}\left(\left(D_{x_0-;g}^{\alpha}f_i\right)\circ g^{-1}\right)(z)\,dz\right)dx\right]. \tag{28}$$

Finally, adding (27) and (28) we obtain the useful identity

$$\theta(f_1,...,f_r)(x_0):=$$

$$\sum_{i=1}^{r}\left[\int_{a}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)f_i(x)\,dx-\left(\int_{a}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)dx\right)f_i(x_0)\right]=$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{\alpha-1}\left(\left(D_{x_0-;g}^{\alpha}f_i\right)\circ g^{-1}\right)(z)\,dz\right)dx\right]\right.$$

$$\left.+\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{\alpha-1}\left(\left(D_{x_0+;g}^{\alpha}f_i\right)\circ g^{-1}\right)(z)\,dz\right)dx\right]\right]. \tag{29}$$

Therefore we get that

$$\|\theta(f_1,...,f_r)(x_0)\|=$$

$$\left\|\sum_{i=1}^{r}\left[\int_{a}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)f_i(x)\,dx-\left(\int_{a}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)dx\right)f_i(x_0)\right]\right\|\leq \tag{30}$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\left\|\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{\alpha-1}\left(\left(D_{x_0-;g}^{\alpha}f_i\right)\circ g^{-1}\right)(z)\,dz\right)dx\right]\right\|\right.$$

$$\left.+\left\|\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{\alpha-1}\left(\left(D_{x_0+;g}^{\alpha}f_i\right)\circ g^{-1}\right)(z)\,dz\right)dx\right]\right\|\right]\leq$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\left[\int_{a}^{x_0}\left\|\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{\alpha-1}\left(\left(D_{x_0-;g}^{\alpha}f_i\right)\circ g^{-1}\right)(z)\,dz\right)\right\|dx\right]\right. \tag{31}$$

$$\left.+\left[\int_{x_0}^{b}\left\|\left(\prod_{\substack{j=1\\ j\neq i}}^{r} f_j(x)\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{\alpha-1}\left(\left(D_{x_0+;g}^{\alpha}f_i\right)\circ g^{-1}\right)(z)\,dz\right)\right\|dx\right]\right]\leq$$

$$\frac{1}{\Gamma(\alpha)}\sum_{i=1}^{r}\left[\left[\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|\right)\left(\int_{g(x)}^{g(x_0)}(z-g(x))^{\alpha-1}\left\|\left(\left(D_{x_0-;g}^{\alpha}f_i\right)\circ g^{-1}\right)(z)\right\|dz\right)dx\right]\right.$$

$$\left.+\left[\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|\right)\left(\int_{g(x_0)}^{g(x)}(g(x)-z)^{\alpha-1}\left\|\left(\left(D_{x_0+;g}^{\alpha}f_i\right)\circ g^{-1}\right)(z)\right\|dz\right)dx\right]\right]=:(*)\,. \tag{32}$$

Hence it holds

$$\|\theta(f_1,...,f_r)(x_0)\|\leq(*)\leq \tag{33}$$

$$\frac{1}{\Gamma(\alpha+1)}\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(\left(D_{x_0-;g}^{\alpha}f_i\right)\circ g^{-1}\right)\right\|\right\|_{\infty,[g(a),g(x_0)]}\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|\right)(g(x_0)-g(x))^{\alpha}dx\right]\right.$$

$$\left.+\left[\left\|\left\|\left(\left(D_{x_0+;g}^{\alpha}f_i\right)\circ g^{-1}\right)\right\|\right\|_{\infty,[g(x_0),g(b)]}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|\right)(g(x)-g(x_0))^{\alpha}dx\right]\right]\leq \tag{34}$$

$$\frac{1}{\Gamma(\alpha+1)}\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(\left(D_{x_0-;g}^{\alpha}f_i\right)\circ g^{-1}\right)\right\|\right\|_{\infty,[g(a),g(x_0)]}\right.\right.$$

$$\left.(g(x_0)-g(a))^{\alpha}\left(\int_{a}^{x_0}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right]+ \tag{35}$$

$$\left.\left[\left\|\left\|\left(\left(D_{x_0+;g}^{\alpha}f_i\right)\circ g^{-1}\right)\right\|\right\|_{\infty,[g(x_0),g(b)]}(g(b)-g(x_0))^{\alpha}\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\ j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right]\right],$$

proving (20). ☑

Next comes an L_1 estimate.

Theorem 4.2. *All as in Theorem 4.1, with $\alpha \geq 1$. Then*

$$\|\theta(f_1,...,f_r)(x_0)\| \leq \frac{1}{\Gamma(\alpha)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{x_0-;g}^{\alpha}f_i\right)\circ g^{-1}\right\|\right\|_{L_1([g(a),g(x_0)])}\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)(g(x_0)-g(x))^{\alpha-1}dx\right]\right.$$

$$\left.+\left[\left\|\left\|\left(D_{x_0+;g}^{\alpha}f_i\right)\circ g^{-1}\right\|\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)(g(x)-g(x_0))^{\alpha-1}dx\right]\right]. \tag{36}$$

Proof. Next we use that $\alpha \geq 1$. We observe that

$$(*) \overset{(33)}{\leq} \frac{1}{\Gamma(\alpha)}$$

$$\sum_{i=1}^{r}\left[\left[\left\|\left\|\left(D_{x_0-;g}^{\alpha}f_i\right)\circ g^{-1}\right\|\right\|_{L_1([g(a),g(x_0)])}\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)(g(x_0)-g(x))^{\alpha-1}dx\right]\right.$$

$$\left.+\left[\left\|\left\|\left(D_{x_0+;g}^{\alpha}f_i\right)\circ g^{-1}\right\|\right\|_{L_1([g(x_0),g(b)])}\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)(g(x)-g(x_0))^{\alpha-1}dx\right]\right], \tag{37}$$

proving Theorem 4.2. ☑

An L_p estimate follows.

Theorem 4.3. *All as in Theorem 4.1. Let now $p,q>1 : \frac{1}{p}+\frac{1}{q}=1$, with $\alpha > \frac{1}{q}$. Then*

$$\|\theta(f_1,...,f_r)(x_0)\| \leq \frac{1}{(p(\alpha-1)+1)^{\frac{1}{p}}\Gamma(\alpha)}$$

$$\sum_{i=1}^{r}\left[\left\|\left\|\left(D_{x_0-;g}^{\alpha}f_i\right)\circ g^{-1}\right\|\right\|_{q,[g(a),g(x_0)]}\left(\int_a^{x_0}(g(x_0)-g(x))^{\alpha-\frac{1}{q}}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right.$$

$$+\left\|\left\|\left(D_{x_0+;g}^{\alpha}f_i\right)\circ g^{-1}\right\|\right\|_{q,[g(x_0),g(b)]} \tag{38}$$

$$\left.\left(\int_{x_0}^{b}(g(x)-g(x_0))^{\alpha-\frac{1}{q}}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right] =: \Phi(x_0).$$

Proof. We have that

$$(*) \overset{(33)}{\leq} \frac{1}{\Gamma(\alpha)} \sum_{i=1}^{r} \left[\left[\int_{a}^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) \left(\int_{g(x)}^{g(x_0)} (z - g(x))^{p(\alpha-1)} dz \right)^{\frac{1}{p}} \right.\right.$$

$$\left. \left(\int_{g(x)}^{g(x_0)} \left\| \left(\left(D_{x_0-;g}^{\alpha} f_i \right) \circ g^{-1} \right)(z) \right\|^q dz \right)^{\frac{1}{q}} dx \right] +$$

$$\left[\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) \left(\int_{g(x_0)}^{g(x)} (g(x) - z)^{p(\alpha-1)} dz \right)^{\frac{1}{p}} \right.$$

$$\left.\left. \left(\int_{g(x)}^{g(x_0)} \left\| \left(\left(D_{x_0+;g}^{\alpha} f_i \right) \circ g^{-1} \right)(z) \right\|^q dz \right)^{\frac{1}{q}} dx \right] \right] \leq \tag{39}$$

$$\frac{1}{\Gamma(\alpha)} \sum_{i=1}^{r} \left[\left[\int_{a}^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) \frac{(g(x_0) - g(x))^{\alpha - 1 + \frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| \left\| \left(\left(D_{x_0-;g}^{\alpha} f_i \right) \circ g^{-1} \right) \right\| \right\|_{q,[g(a),g(x_0)]} dx \right] \right.$$

$$\left. + \left[\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) \frac{(g(x) - g(x_0))^{\alpha - 1 + \frac{1}{p}}}{(p(\alpha-1)+1)^{\frac{1}{p}}} \left\| \left\| \left(\left(D_{x_0+;g}^{\alpha} f_i \right) \circ g^{-1} \right)(z) \right\| \right\|_{q,[g(x_0),g(b)]} dx \right] \right]$$

$$= \frac{1}{(p(\alpha-1)+1)^{\frac{1}{p}} \Gamma(\alpha)}$$

$$\sum_{i=1}^{r} \left[\left\| \left\| \left(D_{x_0-;g}^{\alpha} f_i \right) \circ g^{-1} \right\| \right\|_{q,[g(a),g(x_0)]} \left(\int_{a}^{x_0} (g(x_0) - g(x))^{\alpha - \frac{1}{q}} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) dx \right) \right. \tag{40}$$

$$\left. + \left\| \left\| \left(D_{x_0+;g}^{\alpha} f_i \right) \circ g^{-1} \right\| \right\|_{q,[g(x_0),g(b)]} \left(\int_{x_0}^{b} (g(x) - g(x_0))^{\alpha - \frac{1}{q}} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) dx \right) \right],$$

proving Theorem 4.3. ☑

We continue with generalized fractional Grüss-Cebysev type inequalities for several functions over a Banach algebra. A uniform estimate follows.

Theorem 4.4. *Let* $(A, \|\cdot\|)$ *be a Banach algebra,* $0 < \alpha \leq 1$, $f_i \in C^1([a,b], A)$, $i = 1, ..., r \in \mathbb{N} - \{1\}$*;* $g \in C^1([a,b])$*, strictly increasing such that* $g^{-1} \in C^1([g(a), g(b)])$*;* $x_0 \in [a,b] \subset \mathbb{R}$ *and* $\theta(f_1, ..., f_r)(x_0)$ *as in (19). Assume that*

$$M(f_1, ..., f_r) := \max_{i=1,..,r} \left\{ \sup_{x_0 \in [a,b]} \left\| \left\| \left(D^{\alpha}_{x_0-;g} f_i \right) \circ g^{-1} \right\| \right\|_{\infty, [g(a), g(x_0)]}, \right. \tag{41}$$

$$\left. \sup_{x_0 \in [a,b]} \left\| \left\| \left(D^{\alpha}_{x_0+;g} f_i \right) \circ g^{-1} \right\| \right\|_{\infty, [g(x_0), g(b)]} \right\} < \infty.$$

Denote by

$$\Delta(f_1, ..., f_r) := \int_a^b \theta(f_1, ..., f_r)(x)\, dx =$$

$$\sum_{i=1}^{r} \left[(b-a) \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) f_i(x)\, dx \right) - \right.$$

$$\left. \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) dx \right) \left(\int_a^b f_i(x)\, dx \right) \right]. \tag{42}$$

Then it holds

$$\|\Delta(f_1, ..., f_r)\| =$$

$$\left\| \sum_{i=1}^{r} \left[(b-a) \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) f_i(x)\, dx \right) - \right. \right.$$

$$\left. \left. \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) dx \right) \left(\int_a^b f_i(x)\, dx \right) \right] \right\| \leq$$

$$\frac{M(f_1, ..., f_r)(b-a)^2 (g(b) - g(a))^{\alpha}}{\Gamma(\alpha+1)} \left(\sum_{i=1}^{r} \left\| \prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j\| \right\|_{\infty, [a,b]} \right). \tag{43}$$

Proof. We have that

$$\|\Delta(f_1, ..., f_r)\| \leq \int_a^b \|\theta(f_1, ..., f_r)(x_0)\|\, dx_0 \overset{(33)}{\leq}$$

$$\frac{1}{\Gamma(\alpha+1)} \sum_{i=1}^{r} \left[\left[\sup_{x_0 \in [a,b]} \left\| \left\| \left(D^{\alpha}_{x_0-;g} f_i \right) \circ g^{-1} \right\| \right\|_{\infty, [g(a), g(x_0)]} \right. \right.$$

$$\left\{\int_a^b\left[(g(x_0)-g(a))^{\alpha}\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right]dx_0\right\}\Bigg]+ \tag{44}$$

$$\Bigg[\sup_{x_0\in[a,b]}\left\|\left\|\left(D_{x_0+;g}^{\alpha}f_i\right)\circ g^{-1}\right\|\right\|_{\infty,[g(x_0),g(b)]}$$

$$\left\{\int_a^b\left[(g(b)-g(x_0))^{\alpha}\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right)\right]dx_0\right\}\Bigg]\Bigg]\le$$

$$\frac{(g(b)-g(a))^{\alpha}}{\Gamma(\alpha+1)}\sum_{i=1}^{r}\Bigg[\Bigg[\sup_{x_0\in[a,b]}\left\|\left\|\left(D_{x_0-;g}^{\alpha}f_i\right)\circ g^{-1}\right\|\right\|_{\infty,[g(a),g(x_0)]}$$

$$\left\{\int_a^b\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right)dx_0\right\}\Bigg]+ \tag{45}$$

$$\Bigg[\sup_{x_0\in[a,b]}\left\|\left\|\left(D_{x_0+;g}^{\alpha}f_i\right)\circ g^{-1}\right\|\right\|_{\infty,[g(x_0),g(b)]}$$

$$\left\{\int_a^b\left(\int_{x_0}^{b}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right)dx_0\right\}\Bigg]\Bigg]=:(**).$$

Hence it holds

$$(**)\le\frac{M(f_1,...,f_r)(g(b)-g(a))^{\alpha}(b-a)^2}{2\Gamma(\alpha+1)}$$

$$\sum_{i=1}^{r}\left[\left[\sup_{x_0\in[a,b]}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right\|_{\infty,[a,x_0]}\right]+\right.$$

$$\left.\left[\sup_{x_0\in[a,b]}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right\|_{\infty,[x_0,b]}\right]\right]$$

$$\le\frac{M(f_1,...,f_r)(b-a)^2(g(b)-g(a))^{\alpha}}{\Gamma(\alpha+1)}\left(\sum_{i=1}^{r}\left\|\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j\|\right\|_{\infty,[a,b]}\right), \tag{46}$$

proving (43). ☑

Next comes an L_p estimate

Theorem 4.5. *Let* $(A, \|\cdot\|)$ *be a Banach algebra,* $0 < \alpha \leq 1$, $f_i \in C^1([a,b], A)$, $i = 1, ..., r \in \mathbb{N} - \{1\}$; $g \in C^1([a,b])$, *strictly increasing such that* $g^{-1} \in C^1([g(a), g(b)])$. *Let* $p, q > 1 : \frac{1}{p} + \frac{1}{q} = 1$, $\frac{1}{q} < \alpha \leq 1$, $x_0 \in [a,b] \subset \mathbb{R}$. *Assume that*

$$N(f_1, ..., f_r) := \max_{i=1,..,r} \left\{ \sup_{x_0 \in [a,b]} \left\| \left\| \left(D^{\alpha}_{x_0-;g} f_i \right) \circ g^{-1} \right\| \right\|_{L_q([g(a), g(x_0)])}, \right.$$

$$\left. \sup_{x_0 \in [a,b]} \left\| \left\| \left(D^{\alpha}_{x_0+;g} f_i \right) \circ g^{-1} \right\| \right\|_{L_q([g(x_0), g(b)])} \right\} < \infty, \tag{47}$$

and set

$$Z(f_1, ..., f_r) := \max_{i=1,..,r} \left\{ \left\| \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j\| \right) \right\|_{q,[a,b]} \right\}. \tag{48}$$

Then

$$\left\| \sum_{i=1}^{r} \left[(b-a) \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) f_i(x) \, dx \right) - \right. \right.$$

$$\left. \left. \left(\int_a^b \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} f_j(x) \right) dx \right) \left(\int_a^b f_i(x) \, dx \right) \right] \right\| \leq$$

$$\frac{2rpN(f_1, ..., f_r) Z(f_1, ..., f_r)}{(p+1)(p(\alpha-1)+1)^{\frac{1}{P}} \Gamma(\alpha)} (b-a)^{\frac{1}{p}+1} (g(b) - g(a))^{\alpha - \frac{1}{q}}. \tag{49}$$

Proof. Here $\Phi(x_0)$ is as in (38).

Clearly then it holds

$$\Phi(x_0) \leq \frac{N(f_1, ..., f_r)}{(p(\alpha-1)+1)^{\frac{1}{p}} \Gamma(\alpha)}$$

$$\sum_{i=1}^{r} \left[\left(\int_a^{x_0} (g(x_0) - g(x))^{\alpha - \frac{1}{q}} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) dx \right) + \right. \tag{50}$$

$$\left. \left(\int_{x_0}^b (g(x) - g(x_0))^{\alpha - \frac{1}{q}} \left(\prod_{\substack{j=1 \\ j \neq i}}^{r} \|f_j(x)\| \right) dx \right) \right].$$

Therefore we obtain

$$\|\Delta(f_1, ..., f_r)\| \leq \int_a^b \|\theta(f_1, ..., f_r)(x_0)\| \, dx_0 \leq$$

$$\int_a^b \Phi(x_0)\,dx_0 \leq \frac{N(f_1,...,f_r)}{(p(\alpha-1)+1)^{\frac{1}{p}}\Gamma(\alpha)}$$

$$\sum_{i=1}^{r}\left[\left[\int_a^b\left(\int_a^{x_0}(g(x_0)-g(x))^{\left(\frac{p(\alpha-1)+1}{p}\right)}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right)dx_0\right]\right.$$

$$\left.+\left[\int_a^b\left(\int_{x_0}^b(g(x)-g(x_0))^{\left(\frac{p(\alpha-1)+1}{p}\right)}\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j(x)\|\right)dx\right)dx_0\right]\right]\leq$$

$$\frac{N(f_1,...,f_r)}{(p(\alpha-1)+1)^{\frac{1}{p}}\Gamma(\alpha)}\sum_{i=1}^{r}\left[\left\|\left(\prod_{\substack{j=1\\j\neq i}}^{r}\|f_j\|\right)\right\|_{q,[a,b]}\right.$$

$$\left[\int_a^b\left(\int_a^{x_0}(g(x_0)-g(x))^{(p(\alpha-1)+1)}dx\right)^{\frac{1}{p}}dx_0\right]+ \tag{51}$$

$$\left.\left[\int_a^b\left(\int_{x_0}^b(g(x)-g(x_0))^{(p(\alpha-1)+1)}dx\right)^{\frac{1}{p}}dx_0\right]\right]=:(\eta).$$

Hence we get

$$(\eta)\leq\frac{rN(f_1,...,f_r)Z(f_1,...,f_r)}{(p(\alpha-1)+1)^{\frac{1}{p}}\Gamma(\alpha)}$$

$$\left[\left[\int_a^b\left(\int_a^{x_0}(g(x_0)-g(x))^{(p(\alpha-1)+1)}dx\right)^{\frac{1}{p}}dx_0\right]+\right.$$

$$\left.\left[\int_a^b\left(\int_{x_0}^b(g(x)-g(x_0))^{(p(\alpha-1)+1)}dx\right)^{\frac{1}{p}}dx_0\right]\right]\leq \tag{52}$$

$$\frac{2rpN(f_1,...,f_r)Z(f_1,...,f_r)}{(p+1)(p(\alpha-1)+1)^{\frac{1}{P}}\Gamma(\alpha)}(g(b)-g(a))^{\alpha-\frac{1}{q}}(b-a)^{\frac{1}{p}+1},$$

proving (49). ☑

5. Applications

Remark 5.1. Assume from now on that $(A,\|\cdot\|)$ is a commutative Banach algebra. Then, we get that

$$\theta(f_1,...,f_r)(x_0)\overset{(19)}{=}r\int_a^b\left(\prod_{j=1}^{r}f_j(x)\right)dx-\sum_{i=1}^{r}\left(\int_a^b\left(\prod_{\substack{j=1\\j\neq i}}^{r}f_j(x)\right)dx\right)f_i(x_0), \tag{53}$$

$x_0 \in [a,b]$.

Furthermore, it holds ($0 < \alpha \leq 1$ case)

$$\Delta(f_1,...,f_r) \overset{(42)}{=} r(b-a)\int_a^b \left(\prod_{j=1}^r f_j(x)\right) dx -$$
$$\sum_{i=1}^r \left[\left(\int_a^b \left(\prod_{\substack{j=1\\j\neq i}}^r f_j(x)\right) dx\right)\left(\int_a^b f_i(x)\,dx\right)\right]. \quad (54)$$

When $r = 2$, we get that

$$\theta(f_1,f_2)(x_0) = 2\int_a^b f_1(x) f_2(x)\,dx - f_1(x_0)\int_a^b f_2(x)\,dx - f_2(x_0)\int_a^b f_1(x)\,dx, \quad (55)$$

$x_0 \in [a,b]$,

and

$$\Delta(f_1,f_2) = 2\left[(b-a)\int_a^b f_1(x) f_2(x)\,dx - \left(\int_a^b f_1(x)\,dx\right)\left(\int_a^b f_2(x)\,dx\right)\right], \quad (56)$$

$0 < \alpha \leq 1$.

Corollary 5.2. *All as in Theorem 4.1, A is a commutative Banach algebra, $r = 2$. Then*

$$\|\theta(f_1,f_2)(x_0)\| \leq \frac{1}{\Gamma(\alpha+1)} \sum_{i=1}^2 \left[\left[\left\|\left\|\left\|\left(\left(D_{x_0-;g}^{\alpha} f_i\right)\circ g^{-1}\right)\right\|\right\|\right\|_{\infty,[g(a),g(x_0)]}\right.\right.$$
$$\left.(g(x_0)-g(a))^{\alpha}\left(\int_a^{x_0}\left(\prod_{\substack{j=1\\j\neq i}}^2 \|f_j(x)\|\right) dx\right)\right] +$$
$$\left.\left[\left\|\left\|\left\|\left(\left(D_{x_0+;g}^{\alpha} f_i\right)\circ g^{-1}\right)\right\|\right\|\right\|_{\infty,[g(x_0),g(b)]}(g(b)-g(x_0))^{\alpha}\left(\int_{x_0}^b\left(\prod_{\substack{j=1\\j\neq i}}^2 \|f_j(x)\|\right) dx\right)\right]\right]. \quad (57)$$

Proof. By Theorem 4.1. ☑

We continue with

Corollary 5.3. *All as in Theorem 4.4, A is a commutative Banach algebra, $r = 2$, $0 < \alpha \leq 1$, $M(f_1,f_2)$ as in (41). Then*

$$\|\Delta(f_1,f_2)\| \leq \frac{M(f_1,f_2)(b-a)^2(g(b)-g(a))^{\alpha}}{\Gamma(\alpha+1)}\left[\|\|\|f_1\|\|\|_{\infty,[a,b]} + \|\|\|f_2\|\|\|_{\infty,[a,b]}\right]. \quad (58)$$

Proof. By Theorem 4.4. ☑

Finally we derive:

Corollary 5.4. *All as in Corollary 5.2, for $g(t) = e^t$. Then*

$$\|\theta(f_1, f_2)(x_0)\| \leq \frac{1}{\Gamma(\alpha+1)} \sum_{i=1}^{2} \Bigg[\Bigg[\left\|\left\|\left\|\left(\left(D_{x_0-;e^t}^{\alpha} f_i\right) \circ \log\right)\right\|\right\|\right\|_{\infty,[e^a,e^{x_0}]}$$

$$\left(e^{x_0} - e^a\right)^{\alpha} \left(\int_a^{x_0} \left(\prod_{\substack{j=1 \\ j \neq i}}^{2} \|f_j(x)\|\right) dx\right)\Bigg] +$$

$$\left[\left\|\left\|\left\|\left(\left(D_{x_0+;e^t}^{\alpha} f_i\right) \circ \log\right)\right\|\right\|\right\|_{\infty,[e^{x_0},e^b]} \left(e^b - e^{x_0}\right)^{\alpha} \left(\int_{x_0}^{b} \left(\prod_{\substack{j=1 \\ j \neq i}}^{2} \|f_j(x)\|\right) dx\right)\right]\Bigg]. \tag{59}$$

Proof. By Corollary 5.2. ☑

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