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# Coadjoint semi-direct orbits and Lagrangian families with respect to Hermitian form

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**Abstract.** We use the underlying structure of the coadjoint orbits of a semi-direct product of a connected Lie group and a vector space to obtain families of Lagrangian submanifolds in the adjoint orbits of complex semisimple Lie groups with respect to the symplectic hermitian form. This construction is a generalization of a type of semi-direct orbit previously studied by the authors.

**Keywords**: Coadjoint orbits, Homogeneous spaces, Lagrangian submanifolds, Hermitian symplectic form.

**MSC2020**: 14M15, 22F30, 53D12.

# Órbitas coadjuntas semi-directas y familias Lagrangianas con respecto a la forma Hermitiana

**Resumen.** Sirviendonos de la estructura subyacente de las órbitas coadjuntas del producto semi-directo de un grupo de Lie conexo y un espacio vectorial, construimos familias de subvariedades Lagrangianas en las órbitas adjuntas de un grupo de Lie complejo semisimple con respecto a la forma simpléctica hermitiana. Esta construcción es una generalización de un tipo de órbita semi-directa estudiada previamente por los autores.

**Palabras clave**: Órbitas coadjuntas, Espacios homogéneos, Subvariedades Lagrangianas, Forma simpléctica Hermitiana.

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## 1. Introduction

The symplectic structures and Lagrangian submanifolds of coadjoint orbits were studied and developed by renowned mathematicians such as Kirillov, Arnold, Kostant, and Souriau from the early to the mid-1960s, although they had its roots in the work of Lie, Borel, and Weyl. Alternatively, there are several theories and applications to physics using general reduction theory, as in [2], [9], [10], and [11], among others. We study some applications of the semisimple Lie theory to symplectic geometry, in particular to find Lagrangian submanifolds on adjoint orbits. In this paper, we follow the next construction: let $\mathfrak{g}$ be a non-compact semisimple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$ and Iwasawa decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ with $\mathfrak{a} \subset \mathfrak{s}$ maximal Abelian. In the underlying vector space $\mathfrak{g}$, there is another Lie algebra structure $\mathfrak{k}_{ad} = \mathfrak{k} \times_{ad} \mathfrak{s}$ given by the semi-direct product defined by the adjoint representation of $\mathfrak{k}$ in $\mathfrak{s}$, which is viewed as an Abelian Lie algebra. Let $G = \mathrm{Aut}_0\mathfrak{g}$ be the adjoint group of $\mathfrak{g}$ (identity component of the automorphism group) and put $K = \exp \mathfrak{k} \subset G$. The semi-direct product $K_{ad} = K \times_{ad} \mathfrak{s}$ obtained by the adjoint representation of $K$ in $\mathfrak{s}$ has Lie algebra $\mathfrak{k}_{\mathrm{ad}} = \mathfrak{k} \times_{ad} \mathfrak{s}$, that orbit was studied in [1]. Then, we consider coadjoint orbits for both Lie algebras $\mathfrak{g}$ and $\mathfrak{k}_{ad}$. These orbits are submanifolds of $\mathfrak{g}^*$ that we identify with $\mathfrak{g}$ via the Cartan–Killing form of $\mathfrak{g}$, so that the orbits are seen as submanifolds of $\mathfrak{g}$. These are just the adjoint orbits for the Lie algebra $\mathfrak{g}$ while for $\mathfrak{k}_{ad}$ they are the orbits in $\mathfrak{g}$ of the representation of $K_{ad}$ obtained by transposing its coadjoint representation. The orbits through $H \in \mathfrak{g}$ are denoted by $ad\,(G) \cdot H$ and $K_{ad} \cdot H$, respectively.
In Section 2, our goal is to generalize that construction, to get a wider variety of Lagrangian submanifolds of any adjoint semisimple orbit. For this, we are going to change the usual structure of semisimple Lie algebras, i.e., with a new Lie bracket given by a convenient semi-direct product. This construction was inspired by [8], where the author defines a semi-direct product using a closed subgroup of a semi-simple Lie group and the vector space $\mathfrak{g}$ (seeing $\mathfrak{g} = \mathrm{Lie}(G)$ as a vector space), but focused on solving some applications of control theory. In this way, the first part of this chapter is focused on the general construction of coadjoint orbits of this semi-direct structure. After that, we adapt those general results to the mentioned semi-direct product given by a Cartan decomposition.

In Section 3, we build some families of Lagrangian submanifolds on $ad(G) \cdot H$ with respect to the Hermitian symplectic form $\Omega^{\tau}$, characterized by Cartan involutions on $\mathfrak{g}$. The idea and future goal of this result is: Classify the families of Lagrangian submanifolds determined by Cartan involutions.

## 2. General construction

The construction presented in this section is a more general version of the one given in [1], some proofs will have a lot of similarities, but here we will not be able to use some special structure such as compactness. In Section 2.1, we will see under what conditions the construction is identical to the one cited above.

Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}$ and take a representation $\rho : G \to \mathrm{Gl}\,(V)$ on a vector space $V$ (with $\dim V < \infty$). The infinitesimal representation of $\mathfrak{g}$ on $\mathfrak{gl}(V)$ is also going to be denoted by $\rho$. The vector space $V$ can be seen as an Abelian Lie group (or Abelian Lie algebra). In this way, we can take the semi-direct product $G \times_\rho V$ which is a Lie group whose underlying manifold is the cartesian product $G \times V$. This group is going to be denoted by $G_\rho$ and its Lie algebra $\mathfrak{g}_\rho$ is the semi-direct product

$$\mathfrak{g}_\rho = \mathfrak{g} \times_\rho V.$$

Our first purpose is to describe the coadjoint orbit on the dual $\mathfrak{g}_\rho^*$ of $\mathfrak{g}_\rho$. To begin with, let's see how to determine the $\rho$-adjoint representation $ad_\rho(X, v)$, where $(X, v) \in \mathfrak{g} \times_\rho V$. Thus, take a basis of $\mathfrak{g} \times V$ denoted by $\mathcal{B} = \mathcal{B}_\mathfrak{g} \cup \mathcal{B}_V$ with $\mathcal{B}_\mathfrak{g} = \{X_1, \ldots, X_n\}$ and $\mathcal{B}_V = \{v_1, \ldots, v_d\}$ basis of $\mathfrak{g}$ and $V$, respectively. On this basis, the matrix of $ad_\rho(X, v)$ is given by

$$[ad_\rho(X, v)]_\mathcal{B} = \begin{pmatrix} ad(X) & 0 \\ A(v) & \rho(X) \end{pmatrix}, \tag{1}$$

where $ad(X)$ is the adjoint representation of $\mathfrak{g}$ while for each $v \in V$, $A(v)$ is the linear map $\mathfrak{g} \to V$ defined by

$$A(v)(X) = \rho(X)(v).$$

The dual space $\mathfrak{g}_\rho^*$ can be identified with $\mathfrak{g}^* \oplus V^*$, where $\mathfrak{g}^*$ is immersed on $(\mathfrak{g} \times V)^*$ by extensions of linear functionals on $\mathfrak{g}$ to $\mathfrak{g} \times V$ by the zero functional on $V$ (in the same way, $V^*$ is immersed on $(\mathfrak{g} \times V)^*$). Therefore, the dual basis of $\mathcal{B}$ is $\mathcal{B}^* = \mathcal{B}_\mathfrak{g}^* \cup \mathcal{B}_V^*$, where $\mathcal{B}_\mathfrak{g}^*$ and $\mathcal{B}_V^*$ are the dual basis of $\mathcal{B}_\mathfrak{g}$ and $\mathcal{B}_V$, respectively. Then, the coadjoint representation $ad_\rho^*(X, v)$, for $(X, v) \in \mathfrak{g}_\rho$, with respect to $\mathcal{B}^*$, is transposed with a negative sign on the off-diagonal term of (1), that is

$$\left[ad_\rho^*(X, v)\right]_{\mathcal{B}^*} = \begin{pmatrix} ad^*(X) & -A(v)^* \\ 0 & \rho^*(X) \end{pmatrix}. \tag{2}$$

In this matrix, $ad^*$ is the coadjoint representation of $\mathfrak{g}$, $\rho^*$ is the dual representation of $\rho$, that is

$$\rho^*(X)\alpha = -\alpha \circ \rho(X), \quad \alpha \in V^*, \ X \in \mathfrak{g},$$

and $A(v)^* : V^* \to \mathfrak{g}^*$ is the transpose of $A(v)$ for $v \in V$, which by the above equation can be seen as follows:

$$A(v)^*(\alpha)(X) = \alpha(A(v)(X)) = \alpha(\rho(X)(v)) = -\rho^*(X)(\alpha)(v).$$

The adjoint representation $ad_\rho$ and coadjoint representation $ad_\rho^*$ of $G_\rho$ are obtained by exponentials of representations in $\mathfrak{g}_\rho$. In particular, the following matrices are obtained (on the basis $\mathcal{B}$ and $\mathcal{B}^*$):

$$\left[e^{t ad_\rho(0,v)}\right]_\mathcal{B} = \begin{pmatrix} 1 & 0 \\ tA(v) & 1 \end{pmatrix}, \qquad \left[e^{t ad_\rho^*(0,v)}\right]_{\mathcal{B}^*} = \begin{pmatrix} 1 & -tA(v)^* \\ 0 & 1 \end{pmatrix}. \tag{3}$$

On the other hand, for $g \in G$ the restriction of $ad_\rho(g)$ to $V$ coincides with $\rho(g)$ and the restriction of $ad_\rho^*(g)$ to $V^*$ coincides with $\rho^*(g)$, where we are seeing $V$ and $V^*$ as subspaces of $\mathfrak{g}_\rho = \mathfrak{g} \oplus V$ and $\mathfrak{g}_\rho^* = \mathfrak{g}^* \oplus V^*$, respectively.

To describe the map $A(v)^*$, it is convenient to define the momentum map of the representation $\rho$.

**Definition 2.1.** The **momentum map of the representation** $\rho$ is the map

$$\mu_\rho = \mu : V \otimes V^* \to \mathfrak{g}^*,$$

given by

$$\mu(v \otimes \alpha)(X) = \alpha(\rho(X)v) \qquad v \in V,\ \alpha \in V^*,\ X \in \mathfrak{g}. \tag{4}$$

Then

$$A(v)^* \alpha = \mu(v \otimes \alpha) \in \mathfrak{g}^*,$$

because we have the following identifications

$$A(v)^*(\alpha)(X) = \alpha(A(v)(X)) = \alpha(\rho(X)(v)) = \mu(v \otimes \alpha)(X).$$

**Lemma 2.2.** *The momentum map is $G$-equivariant, with respect to the representation $\rho \otimes \rho^*$ and the coadjoint representation, i.e., for $g \in G$, $v \in V$, and $\alpha \in V^*$*

$$\mu((\rho(g)v) \otimes (\rho^*(g)\alpha)) = ad^*_\rho(g) \cdot \mu(v \otimes \alpha). \tag{5}$$

*Proof.* Let $g \in G$, note that the restrictions of $ad_\rho(g)$ and $ad^*_\rho(g)$ to $V$ and $V^*$ coincide with $\rho(g)$ and $\rho^*(g)$, respectively. Then, for $X \in \mathfrak{g}$

$$\begin{aligned} ad^*_\rho(g)(\mu(v \otimes \alpha))(X) &= (A(v)^*\alpha)\left(ad_\rho\left(g^{-1}\right)\right)(X) \\ &= \alpha\left(ad_\rho\left(g^{-1}\right)\right)(\rho(X)ad_\rho(g)(v)) \\ &= \mu((\rho(g)v) \otimes (\rho^*(g)\alpha))(X). \quad ☑ \end{aligned}$$

Since $\mu$ is bilinear, for any fixed $\alpha \in V^*$, the map $\mu_\alpha : V \to \mathfrak{g}^*$ given by $\mu_\alpha(v) = \mu(v \otimes \alpha)$ is a linear map and consequently, its image $\mu_\alpha(V)$ is a subspace of $\mathfrak{g}^*$. Let $\alpha \in V^*$, the coadjoint orbit of $G_\rho$ through $\alpha$ will be denoted by

$$G_\rho \cdot \alpha := ad^*_\rho(G_\rho) \cdot \alpha.$$

The following proposition shows that the coadjoint orbit for $\alpha \in V^*$ is the union of subspaces $\mu_\beta(V)$, with $\beta \in \rho^*(G)\alpha$.

**Proposition 2.3.** *For $\alpha \in V^*$, the coadjoint orbit can be written as*

$$G_\rho \cdot \alpha = \bigcup_{\beta \in \rho^*(G)\alpha} \mu_\beta(V) \times \{\beta\} \subset \mathfrak{g}^* \times V^*,$$

*and identifying $\mathfrak{g}^* \times V^*$ with $\mathfrak{g}^* \oplus V^*$*

$$G_\rho \cdot \alpha = \bigcup_{\beta \in \rho^*(G)\alpha} \beta + \mu_\beta(V),$$

*where $\beta + \mu_\beta(V)$ is an affine subspace of $\mathfrak{g}^* \oplus V^*$.*

*Proof.* Firstly, if $g \in G$ we can identify $ad_\rho^*(g)$ with $\rho^*(g)$ in the subspace $V^* \subset \mathfrak{g}^* \times V^*$. Therefore, $\rho^*(G)\,\alpha \subset ad_\rho^*(G_\rho)\,\alpha$, and as we saw before

$$\left[e^{tad_\rho^*(0,v)}\right]_{\mathcal{B}} = \begin{pmatrix} 1 & -tA(v)^* \\ 0 & 1 \end{pmatrix},$$

which shows that if $\beta \in V^* \subset \mathfrak{g}^* \oplus V^* = \mathfrak{g}^* \times V^*$, then

$$e^{tad_{\mathfrak{g}_\rho}^*(0,v)}\beta = \beta - tA(v)^*(\beta),$$

which in terms of the momentum map is

$$\beta - tA(v)^*(\beta) = \beta - \mu_\beta(tv).$$

Then, varying $v \in V$, we can see that the affine subspace $\beta + \mu_\beta(V)$ is contained in the coadjoint orbit of $\beta$, for $\beta \in V^*$. As $\rho^*(G) \cdot \alpha \subset ad_\rho^*(G_\rho) \cdot \alpha$, we conclude that

$$\bigcup_{\beta \in \rho^*(G)\alpha} \beta + \mu_\beta(V) \subset G_\rho \cdot \alpha.$$

Conversely, if $g \in G$ and $\beta \in V^*$

$$\begin{aligned} ad_\rho^*(g)(\beta + \mu_\beta(V)) &= \rho^*(g)\beta + ad_\rho^*(g)\mu_\beta(V) \\ &= \rho^*(g)\beta + \mu_{\rho^*(g)\beta}(V) \end{aligned}$$

where the last equality is a consequence of the fact that $\mu$ is equivariant. For $h \in G_\rho$, there are $g \in G$ and $v \in V$, such that

$$ad_\rho^*(h)\,\alpha = ad_\rho^*(g)\,ad_\rho^*\left(e^{t(0,v)}\right)\alpha.$$

As $ad_\rho^*\left(e^{t(0,v)}\right)\alpha \in \alpha + \mu_\alpha(V)$, then $ad_\rho^*(h)\,\alpha \in \rho^*(g)\,\alpha + \mu_{\rho^*(g)\alpha}(V)$. ☑

The action of $G_\rho$ is obviously transitive on $G_\rho \cdot \alpha$, then it is an homogeneous space given by

$$G_\rho \cdot \alpha = G_\rho / Z_\rho(\alpha) \quad \text{with} \quad Z_\rho(\alpha) = \{(g,v) \in G_\rho : \ (g,v) \cdot \alpha = \alpha\},$$

the isotropy subgroup at $\alpha \in V^* \subset \mathfrak{g}_\rho$, with Lie algebra

$$\mathfrak{z}_\rho(\alpha) = \{(X,v) \in \mathfrak{g}_\rho : \ ad_\rho^*(X,v) \cdot \alpha = 0\}.$$

Then in terms of the basis $\mathcal{B}^*$

$$ad_\rho^*(X,v) \cdot \alpha = -\mu_\alpha(v) + \ \rho^*(X)\alpha,$$

therefore,

$$ad_\rho^*(X,v) \cdot \alpha = 0 \ \Leftrightarrow \ \rho^*(X)\alpha = 0 \ \text{ and } \ \mu_\alpha(v) = 0.$$

Thus

$$\mathfrak{z}_\rho(\alpha) = \{X \in \mathfrak{g} : \ \rho^*(X)\alpha = 0\} \times \ker \mu_\alpha, \tag{6}$$

and

$$Z_\rho(\alpha) = \{g \in G : \ \rho^*(g)\alpha = \alpha\} \times \ker \mu_\alpha. \tag{7}$$

As $T_\alpha(G_\rho \cdot \alpha) \simeq \mathfrak{g}_\rho/\mathfrak{z}_\rho(\alpha)$, for any $(X, v) \in \mathfrak{g}_\rho$, there exists a (unique) vector field $\widetilde{(X, v)}$ induced by $(X, v)$ at $\xi = \beta + \mu_\beta(w) \in G_\rho \cdot \alpha$ (with $\beta = \rho^*(g)\alpha$, $g \in G$, and $w \in V$) given by

$$\widetilde{(X, v)}_\xi = \frac{d}{dt}\left(\exp_\rho t(X, v)\right) \cdot \xi \,\Big|_{t=0} = -ad_\rho^*(X, v) \cdot \xi.$$

Hence

$$T_\alpha(G_\rho \cdot \alpha) = \{-ad_\rho^*(X, v) \cdot \alpha : \ (X, v) \in \mathfrak{g}_\rho\}.$$

By Proposition 2.3, the coadjoint orbit $G_\rho \cdot \alpha$ is the union of vector spaces and fibers over $\rho^*(G)\,x$ of the representation $\rho^*$. This union is disjoint because given $\xi \in (\beta + \mu_\beta(V)) \cap (\gamma + \mu_\gamma(V))$ then

$$\xi = \beta + X = \gamma + Y \qquad X = \mu_\beta(v), \ Y = \mu_\gamma(w)$$

with $X, Y \in \mathfrak{g}$. Since the sum $\mathfrak{g}_\rho^* = \mathfrak{g}^* \oplus V^*$ is direct, it follows that $\beta = \gamma$ and $X = Y$. Therefore there is a fibration

$$G_\rho \cdot \alpha \to \rho^*(G)\,\alpha,$$

such that an element $\xi = \beta + X \in \beta + \mu_\beta(V)$ associates $\beta \in \rho^*(G)\,\alpha$, and its fibers are vector spaces. The following proposition shows that this fibration can be identified with the cotangent space of $\rho^*(G)\,\alpha$.

Let $\phi$ be the map

$$\phi : G_\rho \cdot \alpha \to T^*(\rho^*(G)\,\alpha),$$

such that, for $\beta \in \rho^*(G)\,\alpha$

$$\phi(\beta + \mu_\beta(V)) = T_\beta^*(\rho^*(G)\,\alpha).$$

This implies that the restriction of $\phi$ to a fiber $\beta + \mu_\beta(V)$ is given by a linear isomorphism

$$\mu_\beta(V) \to T_\beta^*(\rho^*(G)\,\alpha).$$

**Theorem 2.4.** *The map $\phi$ is an isomorphism of vector bundles.*

*If $\omega$ is the KKS symplectic form on $G_\rho \cdot \alpha$ and $\widetilde{\omega}$ is the canonical symplectic form on $T^*(\rho(G)\,\alpha)$, then $\phi$ is a symplectic isomorphism of vector bundles, i.e., $\phi^*\widetilde{\omega} = \omega$.*

The proof of this result has several steps, essentially they are as follows:

- The restriction of $\phi$ to a fiber $w + \mu_w(V)$ is given by the isomorphism:

  $$\mu_w(V) \to T_w^*(\rho(G)\,x).$$

  To see that: take $\xi \in G_\rho \cdot \alpha$, there is a unique $\beta \in \rho^*(G)\,\alpha$, such that $\xi \in \mu_\beta(V)$, then there is $v \in V$ with $\xi = \beta + \mu(v \otimes \beta)$. The vector $v \in V$ defines a linear functional $f_v$ on $V^*$, and of course their respective restriction to $T_\beta(\rho^*(G)\,\alpha)$, therefore $f_v \in T_\beta^*(\rho^*(G)\,\alpha)$. Set

  $$\phi(\xi) = f_v \in T_\beta^*(\rho^*(G)\,\alpha), \qquad \xi = \beta + \mu(v \otimes \beta).$$

  A map $\phi$ is a linear injective map and the linear map $\mu(v \wedge w) \mapsto f_v$ is surjective.

- In the coadjoint orbit $G_\rho \cdot \alpha$ we can define the Konstant–Kirillov–Souriau (KKS) symplectic form, denoted by $\omega$ and defined as

$$\omega_\xi\left(\widetilde{(X_1, v_1)}_\xi, \widetilde{(X_2, v_2)}_\xi\right) = \xi \cdot [(X, w), (Y, z)]_\rho \quad (X_j, v_j) \in \mathfrak{g}_\rho,\ \xi \in G_\rho \cdot \alpha,$$

  where $\widetilde{(X, v)} = ad^*_\rho(X, v)$ is the Hamiltonian vector field of the function $H_{(X,v)} : M \to \mathbb{R}$ given by

$$H_{(X,v)}(\xi) = \xi(X, v) \quad (X, v) \in \mathfrak{g}_\rho.$$

  Furthermore, as is known, for the cotangent bundle $T^*\left(\rho^*\left(G\right)\alpha\right)$ we can define the canonical symplectic form $\widetilde{\omega}$.

- The best way to relate these symplectic forms is through the action of the semi-direct product $G_\rho = G \times V$ on the cotangent bundle of $\rho^*\left(G\right)\alpha$. This action is described in Proposition 3.11 (in a general case), the action of $G_\rho$ on $T^*\left(\rho\left(G\right)\alpha\right)$ is Hamiltonian and then it defines a moment map

$$m : T^*\left(\rho^*\left(G\right)\alpha\right) \to \mathfrak{g}^*_\rho.$$

  The construction of $m$ shows that it is the inverse of $\phi$. Moreover, $m$ is equivariant, that is, it interchanges the actions on $T^*\left(\rho\left(G\right)\alpha\right)$ and the adjoint orbit, which implies that $m$ is a symplectic morphism.

### 2.1. Compact case

Let $U$ be a compact connected Lie group with Lie algebra $\mathfrak{u}$ and take a representation $\rho : U \to \mathrm{Gl}(V)$, where $V$ admits a $U$-invariant inner product $\langle \cdot, \cdot \rangle$ when $V$ is a real vector space (a Hermitian inner product when $V$ is a complex vector space).

We will denote by $U_\rho$ the semi-direct Lie group $U \times_\rho V$, with Lie algebra $\mathfrak{u}_\rho = \mathfrak{u} \times_\rho V$. The inner product allows us to identify $V$ with $V^*$ by

$$v \in V \mapsto \langle v, \cdot \rangle \in V^*,$$

and we can also identify $\rho$ with $\rho^*$ by

$$\rho^*(X)(v)(w) = -\langle \rho(X)w, v \rangle \quad v, w \in V,\ X \in \mathfrak{u}.$$

Now, analogously to the discussion for the general case, we can characterize the coadjoint orbit of $U_\rho$ in terms of the momentum map $\mu : V \otimes V \to \mathfrak{u}^*$ given by

$$\mu\left(v \otimes w\right)\left(X\right) = \langle \rho\left(X\right)v, w \rangle, \quad v, w \in V,\ X \in \mathfrak{u}.$$

By construction, $\rho(u)$ is an isometry for all $u \in U$ with respect to the fixed $U$-invariant inner product, then $\rho(X)$ is a skew-symmetric linear map with respect to $\langle \cdot, \cdot \rangle$ for all $X \in \mathfrak{u}$, and we have

$$\mu(v \otimes w)(X) = \langle \rho(X)v, w \rangle = -\langle \rho(X)w, v \rangle = -\mu(w \otimes v)(X),$$

that is, $\mu$ is skew-symmetric. Therefore, the momentum map $\mu$ is defined in the exterior product $\wedge^2 V = V \wedge V$. Furthermore, the compact Lie algebra $\mathfrak{u}$ admits an $ad$-invariant inner product such that we can identify $\mathfrak{u}^*$ with $\mathfrak{u}$, then

$$\mu : V \wedge V \to \mathfrak{u}.$$

Similarly, the dual $\mathfrak{u}^* \times V^*$ of $\mathfrak{u}_\rho = \mathfrak{u} \times V$ is identified by its inner product which is a direct sum of $ad$-invariant inner products of $\mathfrak{u}$ and $V$. In that identification, the coadjoint representation of $\mathfrak{u}$ can be seen as the adjoint representation of $\mathfrak{u}$ because its inner product is $\mathfrak{u}$-invariant, but the inner product of $V$ is not invariant under the adjoint representation of $V$, then the coadjoint representation of that Abelian algebra is the transpose of its adjoint representation. This means that the coadjoint representation of $\mathfrak{u} \times V$ is written in $\mathfrak{u} \times V$ as type matrices on orthonormal bases:

$$ad^*_\rho (X, v) = \begin{pmatrix} ad\,(X) & -A\,(v) \\ 0 & \rho\,(X) \end{pmatrix} \qquad X \in \mathfrak{u},\ v \in V,$$

where for each $v \in V$, $A\,(v) : V \to \mathfrak{u}$ can be identified by $A\,(v)\,(w) = \mu\,(v \wedge w)$.

Then the representations $ad_\rho$ and $ad^*_\rho$ of $U_\rho$ are obtained by exponentials of representations in $\mathfrak{u}_\rho$, take $v \in V \subset \mathfrak{u} \times V$ and by Proposition 2.3

$$U_\rho \cdot v := ad^*_\rho(U_\rho) \cdot v = \bigcup_{w \in \rho(U)v} w + A(w)\,(V)\,.$$

## 2.2. Examples

We will see some examples of semi-direct coadjoint orbits to compare them with the usual orbits. To begin with, take $\rho$ the canonical representation of $\mathfrak{u} = \mathfrak{so}\,(n)$ in $V = (\mathbb{R}^n, \langle\cdot,\cdot\rangle)$. The momentum map with values in $\mathfrak{u}$ is given by

$$\mu\,(v \wedge w)\,(B) = \langle Bv, w\rangle \qquad B \in \mathfrak{so}\,(n)\,,$$

and as we know the invariant inner product on $\mathfrak{so}(n)$ is

$$(A, B) = \mathrm{tr}\ AB^T = -\mathrm{tr}\ AB.$$

To describe the orbit, take the isomorphism $I : \wedge^2 V \to \mathfrak{so}\,(n)$ given by

$$I\,(v \wedge w)\,(x) = \langle v, x\rangle w - \langle w, x\rangle v,$$

which satisfies

$$I\,(v \wedge w)^T = -I\,(v \wedge w) = I\,(w \wedge v)\,.$$

If $A \in \mathfrak{so}(n)$ we have

$$\begin{aligned} I\,(v \wedge w)\left(A^T x\right) &= \langle v, A^T x\rangle w - \langle w, A^T x\rangle v \\ &= \langle Av, x\rangle w - \langle Aw, x\rangle v. \end{aligned}$$

Let $\{e_1, \ldots, e_n\}$ be an orthonormal basis of $V$, then

$$\begin{aligned} \operatorname{tr}\left(I\left(v \wedge w\right) A^T\right) &= \sum_i \langle I\left(v \wedge w\right) A^T e_i, e_i\rangle \\ &= \langle Av, \sum_i \langle w, e_i\rangle e_i\rangle - \langle Aw, \sum_i \langle v, e_i\rangle e_i\rangle \\ &= 2\langle Av, w\rangle. \end{aligned}$$

Therefore identifying $V^*$ with $V$ by $\langle\cdot,\cdot\rangle$, and $\mathfrak{so}(n)^*$ with $\mathfrak{so}(n)$ by $\frac{1}{2}(\cdot,\cdot)$, the momentum map is $\mu(v \wedge w) = I(v \wedge w)$, that is

$$\mu\left(v \wedge w\right)(x) = \langle v, x\rangle w - \langle w, x\rangle v \qquad \mu\left(v \wedge w\right) \in \mathfrak{so}\left(n\right).$$

For simplicity of notation, we will denote $I(w \wedge v)$ for $v, w \in V$ as $v \wedge w$. If $v$ and $w$ are $n \times 1$ column vectors, we have

$$v \wedge w = vw^T - wv^T$$

which is an $n \times n$ matrix.

As we saw above, the coadjoint representation of $\mathfrak{so}(n) \times_\rho \mathbb{R}$ is given by

$$ad_\rho^*\left(B, v\right) = \begin{pmatrix} ad\left(B\right) & -A\left(v\right) \\ 0 & B \end{pmatrix} \qquad B \in \mathfrak{so}\left(n\right),\ v \in \mathbb{R}^n,$$

where for each $v \in \mathbb{R}^n$, $A\left(v\right) : \mathbb{R}^n \to \mathfrak{so}\left(n\right)$ is the map

$$A\left(v\right)\left(w\right) = vw^T - wv^T.$$

The representation $\mathfrak{so}\left(n\right) \times_\rho \mathbb{R}^n$ defines a representation of the semi-direct product $U_\rho = \mathrm{SO}\left(n\right) \times_\rho \mathbb{R}^n$ on $\mathfrak{so}\left(n\right) \times \mathbb{R}^n$ by exponentials (here $\mathfrak{so}\left(n\right) \times \mathbb{R}^n$ is the space where the $U_\rho$-orbit is being identified). As discussed earlier a $U_\rho$-orbit of $v \in \mathbb{R}^n \subset \mathfrak{so}\left(n\right) \times \mathbb{R}^n$ is given by

$$\bigcup_{w \in \mathcal{O}} w + A\left(w\right)\left(\mathbb{R}^n\right) \qquad \mathcal{O} = \mathrm{SO}\left(n\right) \cdot v.$$

In this case, the orbits of $\mathrm{SO}\left(n\right)$ in $\mathbb{R}^n$ are the $(n-1)$-dimensional spheres centered at the origin.

**Example 2.5.** *For $n = 2$, we have that $\mathfrak{so}\left(2\right) \times_\rho \mathbb{R}^2$ is isomorphic with $\mathbb{R}^3$ and for all $w \in \mathbb{R}^2$ the image $A\left(w\right)\left(\mathbb{R}^2\right) = \mathfrak{so}\left(2\right)$, therefore the coadjoint semi-direct orbits are the circular cylinders with the axis on the line generated by $\mathfrak{so}\left(2\right)$ in $\mathfrak{so}\left(2\right) \times \mathbb{R}^2 \approx \mathbb{R}^3$.*

Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{so}(n)$. We can induce the canonical representation of $\mathfrak{h}$ in $\mathbb{R}^n$ as a restriction on $\mathfrak{so}(n)$, then

$$\mu\left(v \wedge w\right)\left(B\right) = \langle Bv, w\rangle \qquad B \in \mathfrak{h},$$

because the inner product of $\mathbb{R}^n$ is invariant by $\mathfrak{h}$. The trace form $-\mathrm{tr}\ AB$ provides (by restriction) an inner invariant product in $\mathfrak{h}$, that allows us to identify $\mathfrak{h}$ with $\mathfrak{h}^*$.

Let $p : \mathfrak{so}(n) \to \mathfrak{h}$ be the orthogonal projection in relation to the trace form. By the identification above of $\mathfrak{h}^*$ and $\mathfrak{h}$ we can define the $\mathfrak{h}$-momentum map

$$\mu_{\mathfrak{h}} : \wedge^2\mathbb{R}^n \to \mathfrak{h} \quad \text{given by} \quad \mu_{\mathfrak{h}} = p \circ \mu,$$

where $\mu$ is the momentum map of $\mathfrak{so}(n)$. Then

$$\mu_{\mathfrak{h}}(v \wedge w) = p(vw^T - wv^T).$$

For $\mathfrak{u}(n)$, we can take a canonical representation in $\mathbb{C}^n = \mathbb{R}^{2n}$ and see $\mathfrak{u}(n)$ as an immersed subalgebra of $\mathfrak{so}(2n)$ by matrices $2n \times 2n$ of the form

$$\begin{pmatrix} A & -B \\ B & A \end{pmatrix} \qquad A^T = -A,\ B^T = B.$$

Then $\mathfrak{h} = \mathfrak{u}(n)$ and $p : \mathfrak{so}(2n) \to \mathfrak{u}(n)$ is the orthogonal projection with respect to the trace form. Hence the momentum map is

$$\mu_{\mathfrak{u}(n)}(z \wedge w) = p(zw^T - wz^T), \quad z, w \in \mathbb{R}^{2n}.$$

## 3. Coadjoint semi-direct orbit given by a Cartan decomposition

In this section, we will see the results of Section 2 in the structure of any semisimple non-compact Lie algebra, determined by a given Cartan decomposition. This structure was studied and described in [1], where the authors made the following: Let $\mathfrak{g}$ be a non-compact semisimple Lie algebra with Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s}$. As $[\mathfrak{k}, \mathfrak{s}] \subset \mathfrak{s}$, the subalgebra $\mathfrak{k}$ can be represented on $\mathfrak{s}$ by the adjoint representation. Then, we can define the semi-direct product $\mathfrak{k}_{ad} = \mathfrak{k} \times \mathfrak{s}$, where $\mathfrak{s}$ can be seen as an Abelian algebra. This is a new Lie algebra structure on the same vector space $\mathfrak{g}$ where the brackets $[X, Y]$ are the same when $X$ or $Y$ are in $\mathfrak{k}$, but the bracket changes when $X, Y \in \mathfrak{s}$. The identification between $\mathfrak{k}_{ad} = \mathfrak{k} \times \mathfrak{s}$ and its dual $\mathfrak{k}^*_{\text{ad}} = \mathfrak{k}^* \times \mathfrak{s}^*$ is given by the inner product $B_\theta(X, Y) = -\langle X, \theta Y\rangle$, where $\langle \cdot, \cdot \rangle$ is the Cartan-Killing form of $\mathfrak{g}$ and $\theta$ is a Cartan involution. If $A \in \mathfrak{k}$, then $ad\,(A)$ is anti-symmetric with respect to $B_\theta$, while $ad\,(X)$ is symmetric for $X \in \mathfrak{s}$. The moment map is given by

$$\mu\,(X \wedge Y)\,(A) = B_\theta\,(ad\,(A)\,X, Y) \qquad A \in \mathfrak{k};\ X, Y \in \mathfrak{s},$$

the second part of that equality is

$$B_\theta\,([A, X]\,, Y) = -B_\theta\,([X, A]\,, Y) = -B_\theta\,(A, [X, Y]) = -\langle A, [X, Y]\rangle$$

because $[X, Y] \in \mathfrak{k}$. Therefore the moment map of the adjoint representation of $\mathfrak{k}$ on $\mathfrak{s}$ is

$$\mu\,(X \wedge Y) = [X, Y] \in \mathfrak{k} \qquad X, Y \in \mathfrak{s},$$

where $[\cdot, \cdot]$ is the usual bracket of $\mathfrak{g}$. Therefore, the coadjoint representation of the semi-direct product $\mathfrak{k} \times \mathfrak{s}$ is given by (in an orthonormal basis)

$$ad^*\,(X, Y) = \begin{pmatrix} ad\,(X) & -A\,(Y) \\ 0 & ad(X) \end{pmatrix} \qquad X \in \mathfrak{k},\ Y \in \mathfrak{s}$$

where for each $Y \in \mathfrak{s}$, $A(Y) : \mathfrak{s} \to \mathfrak{k}$ is the map $A(Y)(Z) = [Y, Z]$.

Let $G$ be a connected semisimple Lie group with Lie algebra $\mathfrak{g}$ and take $K \subset G$ the subgroup given by $K = \langle \exp \mathfrak{k} \rangle$. The semi-direct product of $K$ and $\mathfrak{s}$ will be denoted by

$$K_{ad} = K \times_{ad} \mathfrak{s}.$$

The coadjoint orbit of $\widetilde{X} \in \mathfrak{s} \subset \mathfrak{k} \times \mathfrak{s}$ is the union of the fibers $A(Y)(\mathfrak{s})$ with $Y$ belonging to the $K$-coadjoint orbit of $\widetilde{X}$ in $\mathfrak{s}$. As $A(Y)(Z) = [Y, Z]$, then $A(Y)(\mathfrak{s}) = ad(Y)(\mathfrak{s})$ where $ad$ is the adjoint representation in $\mathfrak{g}$. To detail the coadjoint orbits of the semi-direct product, take a maximal Abelian subalgebra $\mathfrak{a} \subset \mathfrak{s}$. The $ad(K)$-orbits in $\mathfrak{s}$ are passing through $\mathfrak{a}$ are thus the flags on $\mathfrak{g}$. Take a positive Weyl chamber $\mathfrak{a}^+ \subset \mathfrak{a}$. If $H \in \mathrm{cl}(\mathfrak{a}^+)$ then the orbit $ad(K)H$ is the flag manifold $\mathbb{F}_H$. By Proposition 2.4, the $K_{ad}$-orbit in $H \in \mathrm{cl}(\mathfrak{a}^+)$ is diffeomorphic to the cotangent bundle of $\mathbb{F}_H$, thus the $K_{ad}$-orbit itself is the union of the fibers $ad(Y)(\mathfrak{s})$, with $Y \in \mathbb{F}_H$. In conclusion

$$K_{ad} \cdot H = \bigcup_{Y \in \mathbb{F}_H} Y + ad(Y)(\mathfrak{s}). \tag{8}$$

In this union, the fiber over $H$ is $H + ad(H)(\mathfrak{s})$ with $ad(H)(\mathfrak{s}) \subset \mathfrak{k}$. With the notations above this subspace of $\mathfrak{k}$ is given by

$$ad(H)(\mathfrak{s}) = \sum_{\alpha(H) > 0} \mathfrak{k}_\alpha.$$

### 3.3. Hermitian symplectic form and Lagrangian submanifolds

Let $\mathfrak{g}_{\mathbb{C}}$ be a complex semisimple Lie algebra and $\mathfrak{u}$ its compact real form with Cartan involution $\tau$, such that

$$\mathcal{H}_\tau(X, Y) = -\langle X, \tau Y \rangle_{\mathbb{C}} \quad X, Y \in \mathfrak{g}_{\mathbb{C}}$$

is a Hermitian form of $\mathfrak{g}_{\mathbb{C}}$, where $\langle \cdot, \cdot \rangle_{\mathbb{C}}$ is the complex Cartan–Killing form of $\mathfrak{g}_{\mathbb{C}}$.

**Remark 3.1.** To avoid confusion, we have that $\mathfrak{g}^{\mathbb{C}}$ is the complexification of $\mathfrak{g}$ (or realification for $\mathfrak{g}^{\mathbb{R}}$), the complexification will be denoted at the top. While $\mathfrak{g}_{\mathbb{C}}$ will simply be to indicate that it is complex (or real for $\mathfrak{g}_{\mathbb{R}}$) , this is will be denoted at the bottom.

The imaginary part of $\mathcal{H}_\tau$ will be denoted by $\Omega^\tau$, which is

$$\Omega^\tau(\cdot, \cdot) = \mathrm{im}(\mathcal{H}_\tau(\cdot, \cdot)) \tag{9}$$

is a symplectic form on $\mathfrak{g}_{\mathbb{C}}$ (see [13]) and will be called the **symplectic Hermitian** form determined by $\tau$.

Let $G$ be a connected Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. For that, let $\mathfrak{g}_{\mathbb{C}} = \mathfrak{u} \oplus i\mathfrak{u}$ be a Cartan decomposition with Cartan involution $\tau$, for $\mathfrak{g}$ a semisimple complex Lie algebra. If $U \subset G$ is the compact subgroup with Lie algebra $\mathfrak{u}$. Then, we will denote by $U_{ad}$ its respective semi-direct product (described in Section 3 for the general case).

If $H \in \mathfrak{s} = i\mathfrak{u}$, then its semi-direct orbit is denoted by $U_{ad} \cdot H$, given by

$$U_{ad} \cdot H = \bigcup_{Y \in ad(U)H} (Y + ad(Y)(i\mathfrak{u})).$$

**Remark 3.2.** Without loss generality, we can choose $H \in \mathfrak{a}$ or $H \in \mathrm{cl}(\mathfrak{a}^+)$, where $\mathfrak{a} \subset \mathfrak{s}$ is the maximal Abelian subalgebra of $\mathfrak{g}$, or $\mathfrak{a}^+$ their respective positive Weyl chamber (see [12] and [13]).

In [1] it was proved that the form $\Omega^\tau$ of $\mathfrak{g}$ restricted to $U_{ad} \cdot H$ is a symplectic form, for $H \in \mathrm{cl}(\mathfrak{a}^+)$ and the following theorem:

**Theorem 3.3.** *The manifolds* $(U_{ad} \cdot H,\ \Omega^\tau)$ *and* $(ad(G) \cdot H,\ \Omega^\tau)$ *are symplectomorphic.*

### 3.4. Lagrangian families

Let $\mathfrak{g}$ be a real semisimple non-compact Lie algebra, such that is a real form of $\mathfrak{g}_\mathbb{C}$, and $\mathfrak{u}$ a compact real form of $\mathfrak{g}_\mathbb{C}$ with Cartan involution $\tau$ (i.e., $\mathfrak{g}$ and $\mathfrak{u}$ are real forms of $\mathfrak{g}_\mathbb{C}$). Then

$$\mathfrak{g} = \underbrace{(\mathfrak{g} \cap \mathfrak{u})}_{\mathfrak{k}} \oplus \underbrace{(\mathfrak{g} \cap i\mathfrak{u})}_{\mathfrak{s}} \tag{10}$$

is a Cartan decomposition of $\mathfrak{g}$.

**Lemma 3.4.** *The restriction of* $\mathcal{H}_\tau$ *to* $\mathfrak{g}$ *is real.*

*Proof.* For $X, Y \in \mathfrak{g}$, there are $X_1, Y_1 \in \mathfrak{g} \cap \mathfrak{u}$ and $X_2, Y_2 \in \mathfrak{g} \cap i\mathfrak{u}$ such that $X = X_1 + X_2$ and $Y = Y_1 + Y_2$. Then

$$\tau X_1 = X_1, \quad \tau X_2 = -X_2, \quad \tau Y_1 = Y_1, \quad \tau Y_2 = -Y_2.$$

As we have that

$$\mathcal{H}_\tau(X, Y) = -\langle X, \tau Y \rangle_\mathbb{C} = -\langle X_1 + X_2, Y_1 - Y_2 \rangle_\mathbb{C}$$

where $\langle \cdot, \cdot \rangle_\mathbb{C}$ is the Cartan–Killing form of $\mathfrak{g}_\mathbb{C}$, then

$$\mathcal{H}_\tau(X, Y) = -\langle X_1, Y_1 \rangle_\mathbb{C} - \langle X_2, Y_1 \rangle_\mathbb{C} + \langle X_1, Y_2 \rangle_\mathbb{C} + \langle X_2, Y_2 \rangle_\mathbb{C}. \tag{11}$$

However

$$\begin{aligned} \mathcal{H}_\tau(Y, X) &= -\langle Y, \tau X \rangle_\mathbb{C} \\ &= -\langle Y_1 + Y_2, X_1 - X_2 \rangle_\mathbb{C} \\ &= -\overline{\langle X_1 - X_2, Y_1 + Y_2 \rangle_\mathbb{C}} \\ &= -\overline{\langle X_1, Y_1 \rangle_\mathbb{C}} - \overline{\langle X_1, Y_2 \rangle_\mathbb{C}} + \overline{\langle X_2, Y_1 \rangle_\mathbb{C}} + +\overline{\langle X_2, Y_2 \rangle_\mathbb{C}}, \end{aligned}$$

as $\mathcal{H}_\tau$ is an Hermitian form, we have that $\mathcal{H}_\tau(X, Y) = \overline{\mathcal{H}_\tau(Y, X)}$, thus,

$$\langle X_2, Y_1 \rangle_\mathbb{C} = \langle X_1, Y_2 \rangle_\mathbb{C},$$

and by equation (11), we have that

$$\mathcal{H}_\tau(X, Y) = -\langle X_1, Y_1 \rangle_\mathbb{C} + \langle X_2, Y_2 \rangle_\mathbb{C},$$

but $X_1, Y_1, iX_2, iY_2 \in \mathfrak{u}$, and the restriction of $\langle \cdot, \cdot \rangle_\mathbb{C}$ to $\mathfrak{u}$ is negative-definite, we can conclude that $\mathcal{H}_\tau|_\mathfrak{g}$ is real. ☑

**Corollary 3.5.** $\Omega^{\tau}|_{\mathfrak{g}} \equiv 0.$

Moreover, let $G^{\mathbb{C}}$ be a Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$. Therefore, given any submanifold $M$ of $U_{ad} \cdot H$, $M$ must be contained on $\mathfrak{g}$. In fact, $M$ is an isotropic submanifold of $U_{ad} \cdot H$. By Theorem 3.3 the same applies to any submanifold $M$ of $ad_r(G^{\mathbb{C}}) \cdot H$.

**Remark 3.6.** To avoid confusion, in this subsection, we say $G^{\mathbb{C}}$ to specify that this is a complex Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}}$.

Now, our purpose is to apply the last results for a non-trivial immersion on the coadjoint semi-direct orbit to find some Lagrangian submanifolds. With the Cartan decomposition of $\mathfrak{g}$ given in (10), then

$$\mathfrak{k}_{ad} = \mathfrak{k} \times_{ad} \mathfrak{s} \subseteq \mathfrak{u} \times_{ad} i\mathfrak{u} = \mathfrak{u}_{ad}.$$

Take $K = \langle \exp \mathfrak{k} \rangle$, then $K_{ad} \cdot H$ is an immersed submanifold on $U_{ad} \cdot H$, for $H \in \mathfrak{a}$. Moreover,

$$T_x K_{ad} \cdot H \subseteq \mathfrak{k}_{ad} \qquad \forall x \in K_{ad} \cdot H,$$

where $\mathfrak{k}_{ad}$ can be identified with $\mathfrak{g}$ as a vector space and by Corollary 3.5, the restriction of $\mathcal{H}_{\tau}$ to $\mathfrak{g}$ is real, thus,

$$\Omega^{\tau}|_{\mathfrak{k}_{ad}} \equiv 0.$$

Therefore, $K_{ad} \cdot H$ is an isotropic submanifold of $U_{ad} \cdot H$, we want to see that $K_{ad} \cdot H$ is a Lagrangian submanifold of $U_{ad} \cdot H$, as we can see in the following example.

**Example 3.7.** *For* $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R})$, $\mathfrak{k} = \mathfrak{so}(2)$, *and* $\mathfrak{u} = \mathfrak{su}(2)$. *Given*

$$H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathfrak{a},$$

*we have that* $K_{ad} \cdot H$ *(cylinder) is a 2-dimensional isotropic submanifold of* $U_{ad} \cdot H$*, a 4-dimensional manifold.*

*Hence, the cylinder* $K_{ad} \cdot H$ *is a Lagrangian submanifold of* $U_{ad} \cdot H$*.*

Let $\sigma$ be an **anti-linear involutive conjugation** on $\mathfrak{g}_{\mathbb{C}}$, such that $\mathfrak{g}$ is the subspace of fixed points of $\sigma$, that is

$$\mathfrak{g} = \{X \in \mathfrak{g}_{\mathbb{C}} : \sigma(X) = X\}.$$

If we have that $\mathcal{A} := \{X \in U_{ad} \cdot H : \ \sigma(X) = X\}$ coincides with $K_{ad} \cdot H$, then we can conclude that $K_{ad} \cdot H$ is a Lagrangian submanifold of $U_{ad} \cdot H$, with respect to the Hermitian symplectic form, for $H \in \mathfrak{a}$.

As $K_{ad} \cdot H$ is contained on $\mathfrak{g}$ and it is a submanifold of $U_{ad} \cdot H$, we have that $K_{ad} \cdot H \subseteq \mathcal{A}$. For the opposite inclusion, by equation (8) we have that

$$U_{ad} \cdot H = \bigcup_{Y \in ad(U) \cdot H} Y + ad(Y)\,(i\mathfrak{u})\,,$$

then given an element $x \in U_{ad} \cdot H$ implies that

$$x = \underbrace{Y}_{\in i\mathfrak{u}} + \underbrace{[Y, iZ]}_{\in \mathfrak{u}}, \quad \text{where} \quad Y = ad(u) \cdot H,\ u \in U,\ Z \in \mathfrak{u}.$$

As $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{s}$, we have the following possibilities:

- Take $X \in \mathfrak{k}$, then $e^{tX} \in U$

$$\begin{aligned}\sigma(x) &= \sigma\underbrace{\left(ad(e^{tX})\cdot H\right)}_{\in\mathfrak{s}} + \sigma\left(i[ad(e^{tX})\cdot H, Z]\right)\\ &= ad(e^{tX})\cdot H - i\left([\sigma ad(e^{tX})\cdot H, \sigma Z]\right)\\ &= ad(e^{tX})\cdot H - i\left([ad(e^{tX})\cdot H, \sigma Z]\right),\end{aligned}$$

  if $Z \in \mathfrak{k}$, we have that $\sigma(Z) = Z$ and if $Z \in i\mathfrak{s}$, we have that $\sigma(Z) = -Z$, then $\sigma(x) = x$ if and only if $Z \in i\mathfrak{s}$.

  Thus, $x$ is a fixed point if and only if $x \in K_{ad}\cdot H$.

- Take $X \in i\mathfrak{s}$, then $e^{tX} \in U$

$$\begin{aligned}\sigma(x) &= \sigma\underbrace{\left(ad(e^{tX})\cdot H\right)}_{\in i\mathfrak{k}} + \sigma\left(i[ad(e^{tX})\cdot H, Z]\right)\\ &= -ad(e^{tX})\cdot H - i\left([\sigma ad(e^{tX})\cdot H, \sigma Z]\right)\\ &= -ad(e^{tX})\cdot H + i\left([ad(e^{tX})\cdot H, \sigma Z]\right),\end{aligned}$$

  for $Z \in \mathfrak{u}$, we have that $\sigma(x) \neq x$, then in this case it is impossible to have fixed points.

- For any other possible choice of $X \in \mathfrak{u}$, we do not have fixed points because it would be a combination of the cases above.

Therefore, $\mathcal{A} = K_{ad}\cdot H$ and $K_{ad}\cdot H$ is the set of fixed points of $\sigma$, its dimension is half the dimension of $U_{ad}\cdot H$. Hence,

**Proposition 3.8.** *For $H \in \mathfrak{a}$, the coadjoint orbit $K_{ad}\cdot H$ is a Lagrangian submanifold of $U_{ad}\cdot H$, with respect to the Hermitian symplectic form.*

By Theorem 3.3, the $K_{ad}$-coadjoint orbit is symplectomorphic to $G$-adjoint orbit and $U_{ad}$-coadjoint orbit is symplectomorphic to $G^{\mathbb{C}}$-adjoint orbit, with respect to $\Omega^{\tau}$. Then, we can conclude that

**Corollary 3.9.** *For $H \in \mathfrak{a}$, the orbit $ad(G)\cdot H$ is a Lagrangian submanifold of $ad(G^{\mathbb{C}})\cdot H$, with respect to the Hermitian symplectic form.*

Furthermore, the coadjoint orbit $U_{ad}\cdot H$ is invariant by the automorphism of $\mathfrak{u}$, because any automorphism of $\mathfrak{u}$ leaves invariant its Cartan subalgebra (see [12] or [14]). Given $k \in \mathrm{Aut}(\mathfrak{k})$ we know that the $k$-action on $\mathfrak{g}$ leaves invariant the Cartan decomposition of $\mathfrak{g}$, its maximal Abelian subalgebra and $\mathfrak{u}$ (because $\mathfrak{k}$ is contained in $\mathfrak{u}$). If exp is the exponential between the Lie algebra $\mathfrak{u}$ and the Lie group $\mathrm{Aut}(\mathfrak{u})$, then for any $X \in i\mathfrak{s}$ we have that $\mathfrak{g}^{tX} = \exp(tX)\cdot\mathfrak{g}$ is a real form of $\mathfrak{g}^{\mathbb{C}}$ with Cartan decomposition $\mathfrak{g}^{tX} = \mathfrak{k}^{tX}\oplus\mathfrak{s}^{tX}$. Take $G^{tX}$ a Lie group with Lie algebra $\mathfrak{g}^{tX}\subset\mathfrak{u}$, then we can conclude that.

**Theorem 3.10.** *For $X \in i\mathfrak{s}\subset\mathfrak{u}$, there are a Lagrangian family of submanifolds $\{M_{tX}\}$ on $ad(G^{\mathbb{C}})\cdot H$ with respect to the Hermitian symplectic form. For $t \in I$, $M_{tX} = ad(G^{tX})\cdot\widetilde{H}$.*

*In fact, the family of Lagrangian submanifolds is determined by $\mathfrak{g}$, and given by the $i\mathfrak{s}$-conjugated real forms of $\mathfrak{g}$.*

## *Appendix: Representations and symplectic geometry*

Let $M \subset W$ be an immersed submanifold of the vector space $W$ (real, that is, $W = \mathbb{R}^N$). The cotangent bundle $\pi : T^*M \to M$ is provided with the canonical symplectic form $\omega$. Given a function $f : TM \to \mathbb{R}$ denote by $X_f$ the corresponding Hamiltonian field, such that $df(\cdot) = \omega(X_f, \cdot)$. If $\alpha \in W^*$, the height function $f_\alpha : M \to \mathbb{R}$ is given by

$$f_\alpha(x) = \alpha(x)$$

and also denote by $f_\alpha$ its lifting $f_\alpha \circ \pi$ which is constant on the fibers of $\pi$. Denote by $X_\alpha$ the Hamiltonian field of this function. Since $f_\alpha$ is constant in the fibers, the field $X_\alpha$ is vertical and the restriction to the fiber $T_x^*M$ is constant in the direction of the vector $(df_\alpha)_x \in T_x^*M$. Furthermore, if $\alpha, \beta \in W^*$, the vector fields $X_\alpha$ and $X_\beta$ commutes. In terms of the action of Lie groups and algebras, the commutativity $[X_\alpha, X_\beta] = 0$ means that the map $\alpha \mapsto X_\alpha$ is an infinitesimal action of $W^*$, seen as an Abelian Lie algebra. This infinitesimal action can be extended to an action of $W^*$ (seen as an Abelian Lie group because the fields $X_\alpha$ are complete).

Now, let $R : L \to \mathrm{Gl}(W)$ be a representation of the Lie group $L$ on $W$ and take an $L$-orbit given by $M = \{R(g)x : g \in L\}$. The action of $G$ on $M$ lifts to an action in the cotangent bundle $T^*M$ by linearity. If $\mathfrak{l}$ is the Lie algebra of $L$, then the infinitesimal action of $\mathfrak{l}$ in the orbit $M$ is given by the fields $y \in M \mapsto R(X)y$, where $X \in \mathfrak{l}$ and $R(X)$ also denotes the infinitesimal representation associated to $R$. The infinitesimal action of the lifting in $T^*M$ is given by $X \in \mathfrak{l} \mapsto H_X$, where $H_X$ is the Hamiltonian field on $T^*M$, such that the Hamiltonian function is $F_X : T^*M \to \mathbb{R}$ given by

$$F_X(\alpha) = \alpha(R(X)y) \qquad \alpha \in T_y^*M.$$

The actions of $L$ and $W^*$ in $T^*M$ are going to define an action of the semi-direct product $L \times W^*$, defined by the dual representation $R^*$. The action of $L \times W^*$ on $T^*M$ is Hamiltonian in the sense that the corresponding infinitesimal action of $\mathfrak{l} \times W^*$ is formed by Hamiltonian fields. When we have a Hamiltonian action we can define its moment map (See [13, Section 14.4]). In this case, a map

$$m : T^*M \to (\mathfrak{l} \times W^*)^* = \mathfrak{l}^* \times W.$$

In the action on $T^*M$, the field induced by $X \in \mathfrak{l}$ is the Hamiltonian field $H_X$ of the function $F_X(\alpha) = \alpha(R(X)y)$, while the field induced by $\alpha \in W^*$ is the Hamiltonian field of the function $f_\alpha$. So if $\gamma \in T_y^*M$, $y \in M \subset W$ then for $X \in \mathfrak{l}$ and $\alpha \in W^*$

$$m(\gamma)(X) = \gamma(R(X)y) \quad \text{and} \quad m(\gamma)(\alpha) = \alpha(y)$$

The first term coincides with the momentum $\mu : W \otimes W^* \to \mathfrak{l}^*$ of the representation $R$, that is, $m(\gamma) = \mu(y \otimes \overline{\gamma})$ such that the restriction of $\overline{\gamma} \in W^*$ to the tangent space $T_yM$ is equal to $\gamma$. The second term shows that the linear functional $m(\gamma)$ restricted to $W^*$ is exactly $y$. Consequently,

**Proposition 3.11.** *The moment map $m : T^*M \to \mathfrak{l}^* \times W = \mathfrak{l}^* \oplus W$ is given by*

$$m(\gamma_y) = \mu(y \otimes \overline{\gamma}) + y,$$

*where $\gamma_y \in T_y^*M$ and $\overline{\gamma} \in W^*$, is an element whose restriction to $T_yM = \{R(X)y : X \in \mathfrak{l}^*\}$ equals to $\gamma$.*

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