



A really topological treatment of some aspects of Carathéodory's theory of prime ends

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Abstract. A homeomorphism approximation technique is applied to give (1) proofs of some theorems of C. Carathéodory, and (2) a proof of a theorem of N. Rutt. The proofs use only tools from general topology (and are new in that respect), and a generalization of a theorem of Carathéodory is obtained.

Keywords: Carathéodory's Theory of Prime Ends, homeomorphism approximation, homeomorphism construction, indecomposable continuum, Schoenflies Theorem, confluent mapping, monotone mapping.

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Un enfoque realmente topológico de algunos aspectos de la teoría de los extremos primos de Carathéodory

Resumen. Se aplica una técnica de aproximación de homeomorfismos para proporcionar (1) pruebas de algunos teoremas de C. Carathéodory y (2) una prueba de un teorema de N. Rutt. Las pruebas utilizan únicamente herramientas de la topología general (y son nuevas en ese aspecto), y se obtiene una generalización de un teorema de Carathéodory.

Palabras clave: Teoría de los extremos primos de Carathéodory, aproximación de homeomorfismos, construcción de homeomorfismos, continuo indescomponible, Teorema de Schoenflies, función confluyente, función monótona.

1. Introduction

Probably the simplest, most natural approach to constructing a homeomorphism is the following: Partition the spaces that are involved into finite sets, and then assign the sets in the first space to the sets in the second space in the manner desired. These partitions and set assignments give the first approximation to the homeomorphism. To obtain an improved approximation, partition the spaces again, obtaining refinements of the first partitions, and then assign sets to sets again, keeping these assignments consistent with those already made in the first step. Continue this

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process, getting at the n level partitions of the space that refine the partition chosen at the $n - 1$ level, and then assign sets to sets once more, being consistent with the choices already made. The homeomorphism approximation improves with each new level, and the end result (if all this is done with a fair amount of care) is a homeomorphism from the first space to the second which has the properties desired. This idea has been used by many topologists to construct homeomorphisms. See, for example, [4], [5], [15], [16], and [17].

This author has used this approach to construct self homeomorphisms that show that certain indecomposable continua can admit certain types of complicated dynamical behavior. (See [10], [11], [12], [13], and [14].) In this paper we would like to apply this idea in still another way. It can be used to give general topological proofs to some of the basic aspects of Carathéodory's theory of prime ends. Since this theory has been used in a very powerful way by several dynamicists to obtain results about dynamics occurring in the plane ([3], [25], [2], and [20]), it is of interest now to topologists and dynamicists both, and it is perhaps worth noting that it can be justified using nothing more than some elementary facts from plane topology. Along the way we obtain a slight generalization to Carathéodory's main theorem, and then obtain a Schoenflies theorem and a theorem of Rutt [24] as corollaries.

C. Carathéodory's papers [6] and [7] on prime ends appeared in 1913 before there was a modern definition of a topological space. His theory concerns a certain compactification of open, connected sets on a surface. Suppose that E and E° denote the closed and open unit disks in the plane, respectively. If D is an open, bounded, connected, simply connected subset of the plane, then there is a conformal mapping Φ of E° onto D (Riemann mapping theorem). In his first paper on prime ends [6], Carathéodory obtained a conformal Schoenflies theorem: he proved that if the boundary of D in the plane is homeomorphic to a circle, then Φ extends to a homeomorphism of E onto \overline{D} . In his second paper [7], he made no assumptions on the boundary of D , and obtained partial generalizations of this result.

Other treatments of the theory of prime ends using conformal mapping machinery include those given by C. Pommerenke [23] and L. Ahlfors [1]. A treatment of the theory which relies on heavy analytic machinery is given in M. Ohtsuka's book [22]. J. Mather [20] has given a very nice account of the theory which is largely topological, but his proofs make use of some not-strictly-topological structures (polyhedra and piecewise linear mappings, for example). M.H.A Newman [21] proved the Schoenflies theorem using topological methods, but his proofs do not use prime end theory. That is, of course, because he does not have to have it in the locally connected context. The basic construction involved in the treatment in this paper is a generalization of Newman's.

Carathéodory's theory has been used by many different kinds of mathematicians in many different ways. However, its applications in dynamics have brought it to the attention of this author. The power of the theory in this context comes about as follows: Suppose that D is an open, connected, bounded, simply connected plane region. Then D is homeomorphic to an open disk. Its boundary, though, need not be a simple closed curve, nor even anywhere close to being a simple closed curve. It is possible that it doesn't even contain an arc. Then \overline{D} is not generally a closed disk. However, if \widehat{D} denotes the prime end compactification of D and E , as before, denotes a closed disk, then there is a homeomorphism $\beta: \widehat{D} \rightarrow E$ that takes D to E° . Further, if F is a homeomorphism on the plane such that $F(\overline{D}) = \overline{D}$, then $F|D$ extends to a homeomorphism \widehat{F} on \widehat{D} , and there is a homeomorphism G on E such that $\beta\widehat{F} = G\beta$. Thus, there is a tool with which one can relate the dynamics of F on \overline{D} and ∂D to the more well studied and understood dynamics of G on E and ∂E . In particular, if F is orientation preserving, it is now possible to assign a rotation number ρ for $F|\partial D$: Since $\widehat{F}|\partial\widehat{D}$ is just a homeomorphism of a circle to a circle, there is the associated rotation number ρ for \widehat{F} on $\partial\widehat{D}$, and one can assign ρ to be the rotation number for $F|\partial D$ as well. (A note of caution here: ∂D may also be the boundary for $\mathbb{R}^2 \setminus D$.) Putting the point at ∞ in order to obtain the one-point compactification $(\mathbb{R}^2 \setminus D) \cup \{\infty\}$ of $\mathbb{R}^2 \setminus D$ means that $(\mathbb{R}^2 \setminus D) \cup \{\infty\}$ satisfies the conditions required, and it makes sense to speak of the prime end compactification \widehat{C} of $(\mathbb{R}^2 \setminus D) \cup \{\infty\} = C$. Then an extension \widehat{F}_C of $F|C$ to \widehat{C} , and another rotation number ρ' are obtained. It is not generally the case that $\rho = \rho'$. In fact, it seems that the really interesting dynamics and interesting invariant continua come about when $\rho \neq \rho'$. Authors who have applied prime ends in this way include J. Mather [18], [19]; K. Alligood and J. Yorke [2]; M. Barge and R.

Gillette [3]; and R. Walker [25].

Why then bother with yet another proof of the main results in the theory? In addition to the fact that a slight generalization of the theory is possible in this way, that one need only know general topology to understand it, and that a theorem of Rutt [24] is obtained as a corollary, it seems to this author to be worth noting exactly what assumptions, what structures are necessary to a given theory. Sometimes, for example, one of the things dynamicists wonder about is whether or not they can avoid some pathological behavior and pathological sets (pathological from their point of view, anyway) by making sufficient “nice” assumptions on the maps involved. Can they, for example, avoid indecomposable invariant continua by requiring C^∞ diffeomorphisms? Since they are concerned with the long term asymptotic behavior of the system, perhaps it is not terribly surprising that often the answer is “no” (see [9]). But what exactly is because of the topology and what is because of the other structures involved? In the case of the aspects of prime end theory that interest them, the answer is that it can all be put on an entirely topological basis. Another consideration is this: There have been some attempts to generalize prime end theory to higher dimensions, and thus far, it is the understanding of this author, they have not been very successful. Perhaps an approach that uses the least amount of structure possible has more of a chance at allowing generalization than do others.

The fundamental problem in showing that \widehat{D} is topologically equivalent to E is this: Although D is topologically equivalent to E° , and therefore there exists a homeomorphism τ from D onto E° , the homeomorphism τ is not much of an aid in proving the topological equivalence of \widehat{D} and E , for it gives absolutely no control as the respective boundaries of \widehat{D} and E are approached. This problem will be solved by carefully choosing a certain countable dense subset P of ∂D , choosing a countable dense subset Q of ∂E , and then constructing a one-to-one surjection $\beta': D \cup P \rightarrow E^\circ \cup Q$ such that $\beta'|_D: D \rightarrow E^\circ$ is a homeomorphism. In this way, enough control over what happens as the boundary is approached will be obtained so that it will be possible to extend $\beta'|_D: D \rightarrow E^\circ$ to a homeomorphism $\beta: \widehat{D} \rightarrow E$. The basic idea behind the construction of β' is quite simple. As previously explained, we will get a sequence of increasingly improved approximations to the surjection β' . At each level the sets $D \cup P$ and $E^\circ \cup Q$ will be partitioned into finite collections of sets, etc.

However, even though the idea is quite simple, writing down the details in a sufficiently careful way to ensure that a proof really is there is quite tedious. Other authors have run into this problem, too. Mather [20] notes that, “It does not appear to be possible to give a brief account of Carathéodory’s theory which does not rely on some deep theory”. Although this account relies only on some basic facts from plane topology, and the ideas involved are not difficult, it cannot be called “brief” either.

2. Background, definitions, elementary facts.

We give here only the background needed for this exposition. The reader who is not already familiar with prime ends is referred to Mather’s very complete and very nice treatment [20] for more examples, more details, and a deeper understanding.

Suppose D is a connected, open set in the plane. (Actually, any open, connected subset of a surface will do. A *surface* is a Hausdorff topological space which has a countable basis and each point of which is contained in an open set homeomorphic to either the plane or the closed upper half plane.) A *chain* in D is a sequence $\{V_n\}_{n=1}^\infty$ of open connected subsets of D such that (i) $V_1 \supseteq V_2 \supseteq \dots$; (ii) $\partial_D V_n \neq \emptyset$ and connected for each n ; and (iii) $\overline{\partial_D V_n} \cap \overline{\partial_D V_m} = \emptyset$ for $n \neq m$. (Please note that a very important distinction is being made here. For $B \subseteq A$, $\partial_A B$ and $Cl_A B$ denote the boundary and closure of B in A , respectively. If X is the space containing A , then $Cl_X B = \overline{B}$ and $\partial_X B = \partial B$.) The chain $\tau = \{W_n\}_{n=1}^\infty$ is said to *divide* the chain $\sigma = \{V_n\}_{n=1}^\infty$ provided that for each n , there is m such that $W_m \subseteq V_n$. Two chains are *equivalent* if each divides the other. A chain is *prime* if every chain which divides it is equivalent to it.

A *prime point* of D is an equivalence class of prime chains of D . A *prime end* of D is a prime point of D with a representative prime chain $\{V_n\}_{n=1}^\infty$ such that $\partial V_n \cap \partial D \neq \emptyset$ for all n . Let \widehat{D} denote the collection of all prime points of D .

Given an open set W in D and an element P of \widehat{D} , we will say P divides W provided a representative chain $\{V_n\}_{n=1}^\infty$ of P has the property that $V_n \subseteq W$ for some n . Let $\widetilde{W} = \{P \in \widehat{D} \mid P \text{ divides } W\}$. The reader can check that the following occur:

- (1) The collection $\mathbb{W} = \{\widetilde{W} \mid W \text{ is open in } D\}$ defines a basis for a topology on \widehat{D} .
- (2) Convergence in this topology works as follows: $\lim_{n \rightarrow \infty} P_n = P$ iff given a representative chain $\{V_n\}_{n=1}^\infty$ of P , for each m there exists N such that P_n divides V_m for all $n \geq N$.

If P is a prime end of D with representative chain $\{V_n\}_{n=1}^\infty$, the impression of P is $I(P) = \bigcap_{n=1}^\infty \overline{V}_n$. (Since $I(P)$ is the nested intersection of a collection of continua, $I(P)$ is a continuum. Further, since $\{V_n\}_{n=1}^\infty$ is prime, $I(P) \subseteq \partial D$.)

If A is a subset of a topological space X , and p is a point of X , then p is *accessible* from the set A if there exists an arc P in X such that $P \subseteq A \cup \{p\}$ and p is an endpoint of P .

Suppose now that D is a bounded, simply connected, connected, open plane set. Let us list the elementary facts from plane topology that we will be using:

(1) An *open arc* is a set homeomorphic to $(0,1)$. An open arc C in D is a *cross-cut* if ∂C consists of two distinct points in ∂D . Each cross-cut divides D into exactly two open sets, each of which is homeomorphic to an open disk. (In other words, each of these sets is itself a bounded, simply connected, connected, open plane set.)

(2) The set of points of ∂D that are accessible from D are dense in ∂D . (Although not crucial to our arguments, it is worth noting that ∂D (a) can be an indecomposable continuum, and (b) may not contain an arc, even though it does have to be connected. For example, it could be a pseudocircle. Then a theorem of S. Mazurkiewicz tells us that the set of points of an indecomposable continuum in the plane that are accessible from the plane minus the continuum is a first category subset of the indecomposable continuum. Thus, the accessible points of ∂D may well be a rather small subset of the continuum, although a dense subset.)

(3) If A is an arc in D , then every point of A is accessible from $D \setminus A$.

(4) If D is an arcwise connected subset of the plane, then D is simply connected provided that if S is a simple closed curve contained in D , then the bounded component of the complement of S in the plane is contained in D . This is equivalent to saying that if S is a continuous image of a simple closed curve contained in D , then each bounded component of the complement of S in the plane is contained in D .

Note that if $\{V_n\}_{n=1}^\infty$ is a prime chain in D representing a prime end P , there is a sequence of cross-cuts $\{C_n\}_{n=1}^\infty$ such that

- (a) $C_n \subseteq V_n$;
- (b) C_n separates D into two connected, open sets, one of which -call it U_n - is contained in V_n ; and
- (c) $\{U_n\}_{n=1}^\infty$ is a prime chain equivalent to $\{V_n\}_{n=1}^\infty$.

We will say that $\{C_n\}_{n=1}^\infty$ is a *representative sequence of cross-cuts* for the prime end P .

We will need the following facts, the first one is a simple result of general topology and we leave the proof to the reader, the other two are proved in Mather's paper.

Lemma 2.1. *Suppose X is a connected topological space, and A and B are nonempty, connected, open subsets of X whose boundaries are nonempty, connected, and disjoint. Then one of the following is true: $X = A \cup B$, $\overline{A} \subseteq B$, $\overline{B} \subseteq A$, or $\overline{A} \cap \overline{B} = \emptyset$.*

Theorem 2.2 ([19], Lemma 3.2). *Suppose $\sigma = \{V_1, V_2, \dots\}$ and $\tau = \{W_1, W_2, \dots\}$ are two chains of the surface S . Further, suppose that for each open set O that contains the point x of S , there exists n_0 such that $\partial_D V_n \subseteq O$ for $n \geq n_0$, and that $V_i \cap W_j \neq \emptyset$ for all i, j . Then σ divides τ .*

Corollary 2.3 ([19], Corollary 3.3). *If $\sigma = \{V_1, V_2, \dots\}$ is a chain such that $\{\partial_D V_n\}_{n=1}^\infty$ converges to a point in the surface S , then σ is prime.*

If $x \in \overline{D}$, x is a *principal point* of $P \in \widehat{D}$ if there is a representative chain $\{V_n\}_{n=1}^\infty$ for P such that if x_1, x_2, \dots is a sequence with $x_i \in \partial_D V_i$ for each i , then x_1, x_2, \dots converges to x . The set of all principal points is the principal set of P . We will denote the principal set of P by $X(P)$. The principal set $X(P)$ of the prime end P is a subcontinuum of $I(P)$. (It is clear that $X(P) \subseteq I(P)$, but it requires a bit of proof to show that $X(P)$ is a nonempty continuum. See [20].)

Let us caution the reader about one thing: in general, there are no simple relationships between the impressions of prime ends, the principal sets of prime ends, and the prime ends themselves. The impressions do not generally give a collection of mutually disjoint continua in the boundary, for example. For a given prime end P , the impression and principal set are always nonempty, but beyond noting that the impression always contains the principal set, not much can be said. It is possible that one continuum is the impression for two different prime ends, too. For more explanation of these matters, see [20].

If X is a compact metric space and $C = \{c(0), c(1), \dots, c(m)\}$ is a finite cover of X which consists of closed, regular neighborhoods, then C is a tiling iff $c(i) \cap c(j) = \partial c(i) \cap \partial c(j)$ for each $i \neq j$ in $\{0, 1, \dots, m\}$. (A closed neighborhood B is regular if $\overline{B^\circ} = B$. An open neighborhood B is regular if $(\overline{B})^\circ = B$.) If $f: \{0, 1, \dots, m\} \rightarrow \{0, 1, \dots, n\}$ is a surjection, then we will say that f is a pattern on $\{0, 1, \dots, n\}$. If $G = \{g(0), g(1), \dots, g(m)\}$ and $H = \{h(0), h(1), \dots, h(n)\}$ are tilings of the compact metric space X , then the statement that B follows the pattern f in H means that $g(i) \subseteq h(f(i))$ for each $i \in \{0, 1, \dots, m\}$.

Theorem 2.4. *Suppose that X and Y are compact metric spaces and G_1, G_2, \dots and H_1, H_2, \dots are sequences of tilings of X and Y , respectively, such that*

- (1) for each i , $G_i = \{g(i, 0), g(i, 1), \dots, g(i, a(i))\}$ and $H_i = \{h(i, 0), h(i, 1), \dots, h(i, a(i))\}$;
- (2) if $A \subseteq \{0, 1, \dots, a(i)\}$, then $\cap_{j \in A} g(i, j) \neq \emptyset$ iff $\cap_{j \in A} h(i, j) \neq \emptyset$;
- (3) $\lim_i \text{mesh } G_i = \lim_i \text{mesh } H_i = 0$; and
- (4) for each i , G_{i+1} follows the pattern η_i in G_i and H_{i+1} follows η_i in H_i .

Then for $x \in X$, there is an infinite sequence $j(x, 1), j(x, 2), \dots$ of integers such that

- (1) $x \in g(i, j(x, i))$, and
- (2) $\eta_i(j(x, i + 1)) = j(x, i)$.

Further, if we define $T(x) = \cap_{i < \infty} \overline{h(i, j(x, i))}$ for $x \in X$, then $T: X \rightarrow Y$ is a homeomorphism.

Proof. We will not give his proof here. We leave it as an exercise for the reader, or refer him/her to [10, Theorem 1]. ■

A continuous function $f: X \rightarrow Y$ is *confluent* if the preimage of each continuum K in Y is a collection of continua in X each of which maps onto K . The continuous function f is *monotone* if the preimage of each continuum K in Y is a continuum in X . Clearly,

- (1) monotone maps are confluent;
- (2) there are many monotone self maps on connected open plane sets that are not homeomorphisms;
- (3) there are many confluent self maps on connected open plane sets that are not monotone.

3. Statements and proofs of the theorems.

Theorem 3.1. *Suppose D is an open, bounded, connected subset of the plane whose boundary is connected. Then \widehat{D} is homeomorphic to a closed disk. Further, if $F: \overline{D} \rightarrow \overline{D}$ is a continuous surjection that takes D onto D , takes ∂D onto ∂D , and is confluent, then there are*

- (1) a homeomorphism $\beta: \widehat{D} \rightarrow E$, where $E = \{x \in \mathbb{R}^2 \mid \|x\| \leq 1\}$;
- (2) an extension \widehat{F} of $F|D$ to all of \widehat{D} ; and
- (3) a surjection $G: E \rightarrow E$ such that $\beta \widehat{F} = G\beta$.

Proof. Note that the collection of accessible points on ∂D is not only dense, but it is also the case that if O is an open set in the plane that intersects the boundary of D , then each component of $D \cap O$ contains a point accessible from D . Further, given two different accessible points of ∂D , there are two different prime ends associated with those two points. Choose a countable dense subset P of ∂D as follows: For each $n \in \mathbb{N}$, there is a finite cover of ∂D by connected, open sets of diameter less than $1/n$. If O denotes one member of this collection, then $O \cap D$ has at most a countable number of components. For each of these components choose a point in ∂D accessible from the component. Let P_n denote the resulting collection of accessible points. Note that P_n is countable. Let $P = \bigcup_{n=1}^{\infty} P_n$, so that P is a countable dense subset of ∂D with the desired property.

Next choose a countable dense subset Q of ∂E . List both P and Q as follows: $P = \{p_0, p_1, \dots\}$ and $Q = \{q_0, q_1, \dots\}$. Then choose a point p in D° and let $q = (0, 0)$ in E° . Next choose an arc A_0 from p to p_0 , and an arc A_1 from p to p_1 such that $A_0 \cap \partial D = \{p_0\}$, $A_1 \cap \partial D = \{p_1\}$, and $A_0 \cap A_1 = \{p\}$. In E , let B_0 and B_1 denote the intersection of the rays from q through q_0 and q_1 , respectively, with E .

Now choose a closed disk $D_{1/2}$ in D so that

- (1) $p \in D_{1/2}^\circ$; and
- (2) $\partial_D D_{1/2} \cap A_0$ and $\partial_D D_{1/2} \cap A_1$ consist of exactly one point.

Let $E_{1/2} = \{x \in E \mid \|x\| \leq 1/2\}$.

What do we now have? Each of $D \cup P$ and $E^\circ \cup Q$ has been divided by A_0, A_1 , and $D_{1/2}$; and B_0, B_1 , and $E_{1/2}$, respectively, into four regions with the following properties:

- (3) Each region is connected and closed (in $D \cup P$ or $E^\circ \cup Q$).
- (4) The interior of each region is an open disk, topologically.
- (5) The regions in $D \cup P$ are listed as $G_1 = \{g(1, 0), g(1, 1), g(1, 2), g(1, 3)\}$.
- (6) The regions in $E^\circ \cup Q$ are listed as $H_1 = \{h(1, 0), h(1, 1), h(1, 2), h(1, 3)\}$.
- (7) There is a ‘‘natural’’ correspondence between the four regions, and this is reflected in the listings. (By this we mean the following: $g(1, i) \cong h(1, i)$) and if $A \subseteq \{0, 1, 2, 3\}$, then $\bigcap_{n \in A} g(1, i) \neq \emptyset$ iff $\bigcap_{n \in A} h(1, i) \neq \emptyset$.)

Let $C(1) = \{p_0, p_1\}$ and $C'(1) = \{q_0, q_1\}$. Next choose finite subsets $C(2)$ and $C'(2)$ of P and Q as follows:

- (8) $|C(2)| = |C'(2)|$ with $C(2) \subseteq P \setminus C(1)$ and $C'(2) \subseteq Q \setminus C'(1)$;
- (9) $p_2 \in C(2)$ and $q_2 \in C'(2)$;
- (10) $C(2) = \{p_{2,0}, p_{2,1}, \dots, p_{2,m(2)}\}$ and $C'(2) = \{q_{2,0}, q_{2,1}, \dots, q_{2,m(2)}\}$;
- (11) for $i \in \{0, 1, 2, \dots, m(2)\}$ there is an arc $A_{2,i}$ from p to $p_{2,i}$ in D such that $A_{2,i} \cap \partial D = \{p_{2,i}\}$, $A_{2,i} \cap \partial D_{1/2}$ consists of exactly one point, and $A_{2,i} \cap A_0 = \{p\} = A_{2,i} \cap A_1$;
- (12) for $i \in \{0, 1, 2, \dots, m(2)\}$ there is a ray $B_{2,i}$ from q to $q_{2,i}$ in E° such that $B_{2,i} \cap \partial E = \{q_{2,i}\}$, $B_{2,i} \cap E_{1/2}$ consist of exactly one point, and $B_{2,i} \cap B_0 = \{q\} = B_{2,i} \cap B_1$; and
- (13) for $i, j \in \{0, 1, 2, \dots, m(2)\}$, $A_{2,i} \cap A_{2,j} = \{p\}$.

Note that a natural, circular ordering of the points of not only Q , but also of P , is being set up, and the listing and choice of points in $C(2)$ and $C'(2)$ reflects this: Let $A_{1,0} = A_0$ and $A_{1,1} = A_1$. Each arc $A_{2,j}$ chosen splits one of the open disks in the collection of open disks that make up $D \setminus \bigcup \{A_{1,0}, A_{1,1}, A_{2,1}, \dots, A_{2,j-1}\}$ into exactly two open disk. If $A_{k,1}$ and $A_{k',1'}$ denote the arcs bounding the split disk, we may, without ambiguity, consider $p_{2,j}$ to be between $p_{k,1}$ and $p_{k',1'}$. Likewise (and in a simpler manner) $q_{2,j}$ will be chosen so that $q_{2,j}$ is between $q_{k,1}$ and $q_{k',1'}$. Thus,

- (14) $|C(2) \cap g(1, i)| = |C'(2) \cap h(1, i)|$ for $i \in \{0, 1, 2, 3\}$. Having done this, choose closed disks $D_{1/4}$ and $D_{3/4}$ in D such that $p \in D_{1/4}^\circ \subseteq D_{1/4} \subseteq D_{1/2}^\circ \subseteq D_{1/2} \subseteq D_{3/4}^\circ \subseteq D_{3/4} \subseteq D$, and each point of $D_{3/4}$ is within a distance of $1/4$ from ∂D . Let $E_{1/4} = \{x \in E \mid \|x\| \leq 1/4\}$,

$E_{3/4} = \{x \in E \mid \|x\| \leq 3/4\}$. Further, choose the disks $D_{1/4}$ and $D_{3/4}$ so that the boundary of each intersects each of the arcs $A_{i,j}$ exactly in one point. Note that the collection of all arcs $A_{i,j}$ and disks D_r thus chosen partitions $D \cup P$ into a finite collection of relatively closed, connected, simply connected sets that intersect only at their boundaries in $D \cup P$, and that the collection of all rays $B_{i,j}$ and disks E_r thus chosen partitions $E^\circ \cup Q$ into a finite collection of relatively closed, connected, simply connected sets that intersect only at their boundaries in $E^\circ \cup Q$. See Figure 1. Further, there is a natural correspondence between these collections of sets, which we will name $G_2 = \{g(2,0), g(2,1), \dots, g(2, n(2))\}$ and $H_2 = \{h(2,0), h(2,1), \dots, h(2, n(2))\}$ for $D \cup P$ and $E^\circ \cup Q$, respectively, such that

(15) if $\gamma \subseteq \{0, 1, \dots, n(2)\}$, then $\cap_{i \in \gamma} g(2, i) \neq \emptyset$ iff $\cap_{i \in \gamma} h(2, i) \neq \emptyset$ and

(16) each $g(2, i) \cong h(2, i)$.

Also, G_2 refines G_1 and, therefore, follows some pattern η_1 in G_1 . The collection H_2 follows the same pattern η_1 in H_1

Continue the process that has been started, choosing two sequences G_1, G_2, \dots and H_1, H_2, \dots of collections of covers of $D \cup P$ and $E^\circ \cup Q$, respectively; two sequences $C(1), C(2), \dots$ and $C'(1), C'(2), \dots$ of subsets of P and Q , respectively; two sequences $\mathbb{A}(1) = \{A_{1,0}, A_{1,1}\}, \mathbb{A}_2 = \{A_{2,0}, \dots, A_{2, m(2)}\}, \dots$ and $\mathbb{B}(1) = \{B_{1,0}, B_{1,1}\}, \mathbb{B}(2) = \{B_{2,0}, B_{2,1}, \dots, B_{2, m(2)}\}, \dots$ of arcs in $D \cup P$ and rays in $E^\circ \cup Q$, respectively; $\mathbb{D} = \{D_r \mid r \text{ is a dyadic rational in } (0, 1)\}$, a collection of closed disks in D ; and $\mathbb{E} = \{E_r \mid r \text{ is a dyadic rational in } (0, 1)\}$, where $E_r = \{x \in E \mid \|x\| \leq r\}$, so that

(17) G_{i+1} follows the pattern η_i in G_i , and H_{i+1} follows the pattern η_i in H_i ;

(18) we list $G_i = \{g(i,0), g(i,1), \dots, g(i, n(i))\}$ and $H_i = \{h(i,0), h(i,1), \dots, h(i, n(i))\}$;

(19) each member of P is in some $C(i)$ and each member of Q is in some $C'(i)$;

(20) each $C(i)$ is finite and $|C(i)| = |C'(i)|$;

(21) each arc $A_{i,j}$ intersects ∂D in exactly the point $p_{i,j}$ in $C(i) = \{p_{i,1}, \dots, p_{i, m(i)}\}$, intersects the boundary of the disk D_r in exactly one point, and intersects any other arc $A_{k,1}$ in exactly the point p ;

(22) $\cup D_r = D$ with each point of the boundary of $D_{(2^n-1)/2^n}$ within a distance of $1/2^n$ from ∂D ;

(23) each ray $B_{i,j}$ intersects ∂E in exactly the point $q_{i,j}$; and

(24) if r and r' are dyadic rationals in $(0,1)$ and $r < r'$, then $D_r \subseteq D_{r'}$.

We may also, without loss of generality, require that the following properties hold:

(25) $\lim_{i \rightarrow \infty} \text{mesh} \{g(i, j) \in G_i \mid g(i, j) \subseteq D_r\} = 0$, for r a fixed dyadic rational in $(0,1)$.

(26) $\lim_{i \rightarrow \infty} \text{mesh} \{g(i, j) \cap A_{k,i} \mid g(i, j) \in G_i\} = 0$ (for fixed $A_{k,1}$).

Finally then, we may define a map $\beta: P \cup D \rightarrow Q \cup E^\circ$ as follows: If $x \in D \cup P$, then there is an infinite sequence $j(x, 1), j(x, 2), \dots$ of integers such that for each i

(27) $x \in g(i, j(x, i))$, and

(28) $\eta_i(j(x, i+1)) = j(x, i)$. Define $\beta(x) = \cap_{i \in \mathbb{N}} h(i, j(x, i))$. It is not difficult to check that β is well-defined, one-to-one, and onto. Further, because of the fact that if $X = D_r$ for some allowable r , or $X = A_{k,l}$ for some allowable k and l , then $G_i(X) = \{g(i, j) \cap X \mid j \in \{0, \dots, n(i)\}\}$ and $g(i, j) \cap X \neq \emptyset\}$ is a tiling on X , and $H_i(X) = \{h(i, j) \cap \beta(X) \mid (g(i, j) \cap X) \in G_i(X)\}$ is a tiling on $\beta(X)$, and the fact that the tilings involved satisfy the requirements of the background theorem given, it follows that $\beta|_X: X \rightarrow \beta(X)$ is a homeomorphism. Also, $\beta(g(i, j)) = h(i, j)$, $\beta(A_{k,l}) = B_{k,l}$, $\beta(D_r) = E_r$, and $\beta(p_{k,1}) = q_{k,1}$ (for appropriate i, j, k, l , and r). Then $\beta|_D: D \rightarrow E^\circ$ is a homeomorphism.

β is also continuous: Suppose $q_{m,1} \in Q$ and O is an open set in E that contains $q_{m,1}$. Then there is some $h(i, j)$ in some H_i such that $q_{m,1} \in h(i, j)^\circ$ and $h(i, j) \subseteq O$. Since $p_{m,1} \in g(i, j)^\circ$ and if $x \in g(i, j)$, then $\beta(x) \in h(i, j)$, it follows that β is continuous on $D \cup P$.

We have now a map from D to E° that is "sufficiently nice". That is, if \mathbb{P} is a prime end in \widehat{D} , then \mathbb{P} may be represented by a prime chain $\{V_n\}_{n=1}^\infty$ such that $\overline{\partial_D V_n}$ is an arc whose endpoints are two

points in P , that intersects ∂D only in those two points, and such that $\beta|\overline{\partial_D V_n}$ is a homeomorphism. By construction, $\{\beta(V_n)\}_{n=1}^\infty$ is a prime chain in $\widehat{E}(\cong E)$. If we define, $\tilde{\beta}(\mathbb{P}) = \mathbb{Q}$, the prime end represented by $\{\beta(V_n)\}_{n=1}^\infty$, then $\tilde{\beta}: \widehat{D} \rightarrow \widehat{E}$ is well-defined, and it is a simple matter to check that it is in fact a homeomorphism.

Let us turn our attention then to F . Note that the diagram below commutes:

$$\begin{array}{ccc}
 D & \xrightarrow{F|D} & D \\
 \beta \downarrow & & \downarrow \beta \\
 E^\circ & \xrightarrow{G} & E^\circ
 \end{array}
 \quad (\text{where } G = \beta F \beta^{-1}|E^\circ)$$

Suppose $\{V_n\}_{n=1}^\infty$ is a prime chain in D chosen so that $\lim_n \text{diam}(\partial_D V_n) = 0$, and $\lim_n \partial_D V_n = \{x\}$. Then $\{F(V_n)\}_{n=1}^\infty$ is a nested collection of connected sets in D , as is $\{\overline{F(V_n)}\}_{n=1}^\infty$. For each x in D , let $\epsilon_x = \min\{\|x - y\| \mid y \in \partial D\}/2$, and let $B_\epsilon(z)$ denote the ϵ -ball about the point z in the plane. For each positive integer n , let $W_n = \cup\{B_{\epsilon_x/2n}(x) \mid x \in D \cap F(V_n)\}$. Hence, W_n is connected and open in D , and $\{W_n\}_{n=1}^\infty$ is a chain, but is it a prime chain?

Note that if W is a connected, open set of \overline{D} , then $F^{-1}(W)$ consists of a collection of connected, open sets in \overline{D} , each of which maps onto W . For each n , let K_n denote the component of $F^{-1}(W_n)$ that contains V_n . It must be the case that $Cl_D K_{n+1} \subseteq K_n$. Also, if α is the prime point represented by $\{V_n\}_{n=1}^\infty$, then $F(I(\alpha)) = \cap \overline{F(V_n)} = \cap \overline{W_n}$. Suppose that the chain $\{W'_n\}_{n=1}^\infty$ divides $\{W_n\}_{n=1}^\infty$. Without loss of generality, we may assume that $Cl_D W'_n \subseteq W_n$ and that $F(V_n) \cap W'_n \neq \emptyset$. Let K'_n denote that component of $F^{-1}(W'_n)$ that intersects V_n . Now $\{K'_n\}_{n=1}^\infty$ is a chain in D . But it follows from Mather's theorem (Theorem 2.2) that $\{K'_n\}_{n=1}^\infty$ divides $\{V_n\}_{n=1}^\infty$, from which it follows that each W'_n is contained in some $F(V_m)$. Thus, $\{W_n\}_{n=1}^\infty$ is prime. If ν is that prime point represented by $\{W_n\}_{n=1}^\infty$, define $\widehat{F}(\alpha)$ to be ν . Since every prime point in \widehat{D} may be represented by such a prime chain as $\{V_n\}_{n=1}^\infty$, we have a well defined function from \widehat{D} onto \widehat{D} with \widehat{F} . It is easy to check that it is also continuous. Then if $\widehat{G} = \beta \widehat{F} \beta^{-1}$, we are done. ■

Theorem 3.2. (Schoenflies) *Suppose that D is a bounded, connected, regular, open subset of the plane, that ∂D is locally connected, and that \overline{D} is simply connected. Then \widehat{D} is homeomorphic to a closed disk.*

Proof. Please recall the proof and construction given for Carathéodory's Theorem, as we refer to it here and give the notation used the same meaning.

First, in the countable dense subset P of ∂D that was chosen, a circular ordering was set up on the members of P . In particular, note that whenever $p_{m,1} = a$ and $p_{m',1'} = a'$ are members of P , suppose T denotes an arc in ∂D with endpoints a and a' . Since $A_{m,1} \cup T \cup A_{m',1'} = T'$ is a simple closed curve in \overline{D} , the bounded component C of the complement of T' is contained in D . Let U_1 and U_2 denote the components of $D \setminus (A_{m,1} \cup A_{m',1'})$. Either $C \subseteq U_1$ or $C \subseteq U_2$, say $C \subseteq U_1$. Then $\partial_D C = \partial_D U_1$. Each radial arc $A_{r,s}$, besides $A_{m,1}$ and $A_{m',1'}$, intersects either U_1 or U_2 . Since $\partial_D C = \partial_D U_1$, any radial arc that intersects U_1 also intersects C . Then it is contained, except for its endpoints, in C . It follows that $C = U_1$ (because C is dense in U_1 and U_1 is regular), and $T' = \overline{U_1} \setminus U_1$.

Further, since ∂D is locally connected, and therefore points close together in P may be connected with arcs of small diameter, we may choose a finite subset $\{a(0), a(1), \dots, a(n)\}$ of P , with $a(0) = p_0 < a(1) < \dots < a(n)$, such that each pair $a(i), a(i + 1)$ is connected by an arc $T(i)$ in ∂D whose endpoints are $a(i)$ and $a(i + 1)$, and such that each $T(i)$ contains each point in P "between" $a(i)$ and $a(i + 1)$. Further, we may choose this set so that an arc $T(n)$ in ∂D whose endpoints are $a(n)$ and $a(0)$ contains every point "between" $a(0)$ and $a(n)$ in P . Then $\cup T(i) = \partial D$. Let $C(i)$

denote the bounded component of the complement of the simple closed curve formed by $T(i)$ and its associated radial arcs. Suppose $T(i) \cap T(j) \neq \emptyset$. Without loss of generality, we may assume that $j = i + 1$. (If not, we can combine any intervening arcs, and lose nothing.) Now $T(i) \cap T(i + 1)$ does at least contain the common endpoint $a(i + 1)$, but suppose there is another point in the intersection.

Suppose the point $t \neq a(i + 1)$ is in the intersection, and that $n > 1$. There are points $p_{m,1}, p_{m',1'}$, and $p_{k,r}$ in P such that the ends of $T(i)$ are $p_{m,1}$ and $p_{m',1'}$, and the ends of $T(i + 1)$ are $p_{m',1'}$ and $p_{k,r}$. Now $p_{m,1} < p_{m',1'} < p_{k,r}$. Let $[p_{m,1}, t]$ and $[t, p_{m',1'}]$ denote the subarcs of $T(i)$ determined by t , and let $[p_{m',1'}, t]$ and $[t, p_{k,r}]$ denote the subarcs of $T(i + 1)$ determined by t . (It is possible that one of these "arcs" is degenerate.) Now we may assume that $S(1) = A_{m,1} \cup [p_{m,1}, t] \cup [t, p_{k,r}] \cup A_{k,r}$ is a simple closed curve. However, $[t, p_{m',1'}] \cup [p_{m',1'}, t] = S(2)$ is the continuous image of a simple closed curve which is nondegenerate and intersects $S(1)$ in exactly the point t . But this is impossible: The bounded component of the complement of $S(1)$ contains $C(i) \cup C(i + 1)$, and so $p_{m',1'} \in S(1) \cap S(2)$. However, $p_{m',1'} \notin S(1)$.

By construction, for the case where ∂D is locally connected, $\beta^{-1}: E^\circ \cup Q \rightarrow D \cup P$ is a homeomorphism, for in this situation, $\lim_i \text{mesh } G_i = 0$. In fact, we may go even further: Let $\overline{G}_i = \{\overline{g(i, j)} \mid g(i, j) \in G_i\}$. Then each \overline{G}_i is a tiling of \overline{D} , and we could just as well have constructed a homeomorphism β from \overline{D} to E . (This makes heavy use of the fact that $g(i, j) \cap g(i', j') \neq \emptyset$ iff $\overline{g(i, j)} \cap \overline{g(i', j')} \neq \emptyset$, and that $\partial g(i, j) \cap \partial g(i', j') \cap \partial D = \emptyset$ or consists of one point of P .) ■

Theorem 3.3. (Rutt [24], Theorem 4) *If ∂D is an indecomposable continuum, then there is a prime end P in \widehat{D} such that $I(P) = \partial D$.*

Proof. Again, this proof refers to the construction and notation of the Carathéodory Theorem. For each i , there is some $g(i, j) \in G_i$ such that $\partial g(i, j) \cap \partial D$ is an indecomposable continuum. (Otherwise ∂D is the union of a finite number of nowhere dense subcontinua.) Then we can find an infinite nested sequence $g(1, i_1) \supseteq g(2, i_2) \supseteq \dots$ so that each $g(k, i_k)$ contains ∂D in its boundary in the plane. This sequence $\{g(k, i_k)\}_{k=1}^\infty$ may not be a chain, however, for it is possible that the boundaries of two different members of the sequence intersect in D . We can correct this by considering $r(i) = \{g(i, j) \cup g(i, k) \mid g(i, j) \text{ and } g(i, k) \text{ are adjacent members of } G_i \text{ with points of } \partial D \text{ in their closure}\}$. Then we may obtain an infinite sequence $\{r(i)\}_{i=1}^\infty$ such that for each i and $j \in \mathbb{N}$:

- (1) $\partial D \cap \partial r(i)$ is an indecomposable continuum;
- (2) $Cl_D r(i + 1) \subseteq r(i)$; and
- (3) $\overline{\partial_D r(i)} \cap \overline{\partial_D r(j)} = \emptyset$.

By construction, it also follows that $\{r(i)\}_{i=1}^\infty$ is a prime chain. (Remember the map β and the associated chain $\{\beta(r(i))\}_{i=1}^\infty$.) Thus, $\bigcap_{i=1}^\infty \overline{r(i)} = \partial D = I(P)$, where P is the prime end represented by $\{r(i)\}_{i=1}^\infty$. ■

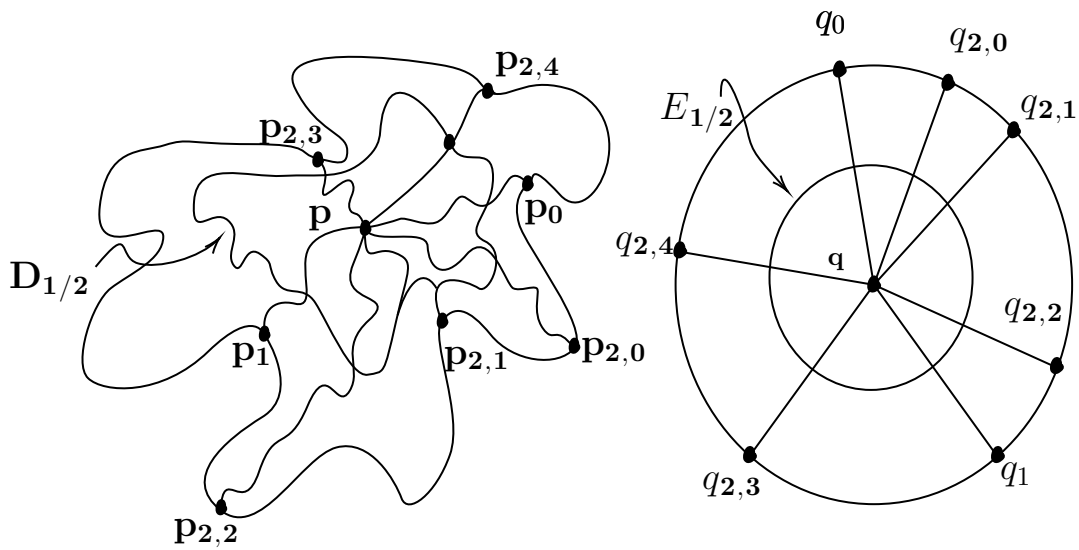


Figure 1

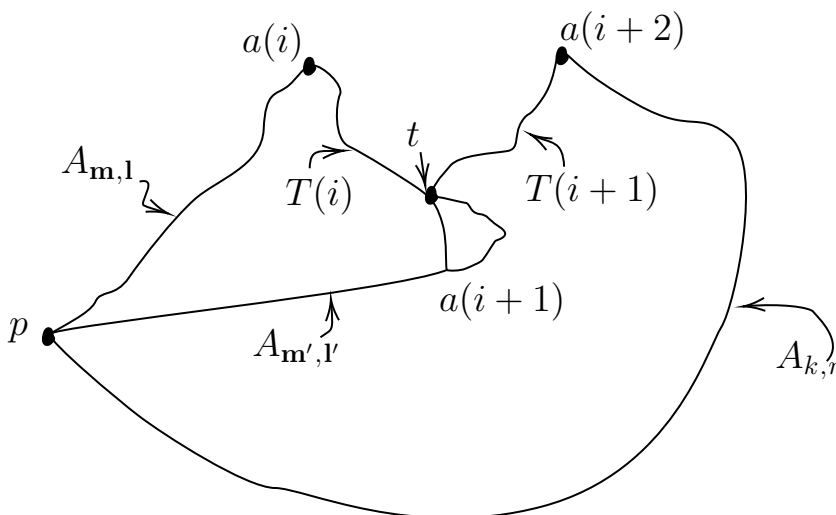


Figure 2
Schoenflies Theorem

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