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An Introduction to Calculus in the $q-$ Real Spinor Variables

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Abstract. In this paper we introduce the calculus in $q-$ real spinor variables. We establish the $q-$ difference operator for $q-$ real spinor variables and the $q-$ spinor real integral formulas. We also define the differential equation on $q-$ real spinor variable, and the suggestions for further work at the end of the paper.

Keywords: $q-$ Spinor real variables, $q-$ differential operators, $q-$ spinor real integral formulas, differential equation in $q-$ spinor variables.

MSC2020: 81Q99, 46E99, 35A24, 15A66, 16T99, 17B37.

Una Introducción al $q-$ Cálculo en la $q-$ Variable Espinorial Real

Resumen. En este artículo introducimos el cálculo en la $q-$ variable espinorial real. Establecemos el $q-$ operador diferencial espinorial y las $q-$ formulas integrales reales spinoriales. También definimos la $q-$ ecuación diferencial en la variable espinorial real y las sugerencias para trabajos futuros al final del artículo.

Palabras clave: $q-$ Variable real spinorial, $q-$ operadores diferenciales, $q-$ formulas integrales reales espinoriales, ecuación diferencial en $q-$ variables espinoriales.

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1. Introduction

Spinor calculus plays a fundamental role in mathematics and physics, with applications in quantum mechanics, general relativity, and field theory. Building upon the foundational works of Berestetskii et al., Lachieze-Rey, Gori et al., and Cartan (see [2],[1],[12],[5] for more details), spinors have been extensively studied in both complex and real-variable frameworks. These studies have provided robust algebraic and geometric tools to describe rotations, reflections, and their corresponding transformation groups, such as $SU(2)$.

Previous works, particularly those by Gori et al [5]. and the author of [8], introduced spinor covariance, contravariance, and extensions into the domain of quantum variables through q- Lorentzian algebra. The earlier research focused primarily on spinor calculus in complex variables, offering a foundation for quantum formulations and algebraic structures such as Clifford algebras.

Definition 1.1. The following are the bosonic q- deformed Minkowskian Pauli spin matrices defined in the Schmidt work [16]:

$$(\sigma^{+})_{\alpha\dot{\beta}} = \begin{bmatrix} 0 & 0 \\ 0 & q \end{bmatrix}, \quad (\sigma^{-})_{\alpha\dot{\beta}} = \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix}, \tag{1}$$

$$(\sigma^{3})_{\alpha\dot{\beta}} = q(q+q^{-1})^{-1/2} \begin{bmatrix} 0 & q^{1/2} \\ q^{-1/2} & 0 \end{bmatrix}, \quad (\sigma^{0})_{\alpha\dot{\beta}} = (q+q^{-1})^{-1/2} \begin{bmatrix} 0 & -q^{-1/2} \\ q^{1/2} & 0 \end{bmatrix}. \tag{2}$$

Likewise, the conjugated Pauli matrices are:

$$(\overline{\sigma}^{+})_{\dot{\alpha}\beta} = \begin{bmatrix} 0 & 0 \\ 0 & q^{-1} \end{bmatrix}, \quad (\overline{\sigma}^{-})_{\dot{\alpha}\beta} = \begin{bmatrix} q^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \tag{3}$$

$$(\overline{\sigma}^{3})_{\dot{\alpha}\beta} = q(q+q^{-1})^{-1/2} \begin{bmatrix} 0 & q^{1/2} \\ q^{-1/2} & 0 \end{bmatrix}, \quad (\overline{\sigma}^{0})_{\dot{\alpha}\beta} = (q+q^{-1})^{-1/2} \begin{bmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{bmatrix}. \tag{4}$$

The inverse Pauli matrices

$$(\sigma_{+}^{-1})_{\alpha\dot{\beta}} = \begin{bmatrix} 0 & 0 \\ 0 & q^{-1} \end{bmatrix}, \quad (\sigma_{-}^{-1})_{\alpha\dot{\beta}} = \begin{bmatrix} q^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \tag{5}$$

$$(\sigma_{3}^{-1})_{\alpha\dot{\beta}} = q(q+q^{-1})^{-1/2} \begin{bmatrix} 0 & q^{1/2} \\ q^{-1/2} & 0 \end{bmatrix}, \quad (\sigma_{0}^{-1})_{\alpha\dot{\beta}} = (q+q^{-1})^{-1/2} \begin{bmatrix} 0 & -q^{-1/2} \\ q^{1/2} & 0 \end{bmatrix}. \tag{6}$$

Finally,

$$(\overline{\sigma}_{+}^{-1})_{\dot{\alpha}\beta} = \begin{bmatrix} 0 & 0 \\ 0 & q^{-1} \end{bmatrix}, \quad (\overline{\sigma}_{-}^{-1})_{\dot{\alpha}\beta} = \begin{bmatrix} q^{-1} & 0 \\ 0 & 0 \end{bmatrix}, \tag{7}$$

$$(\overline{\sigma}_{3}^{-1})_{\dot{\alpha}\beta} = q(q+q^{-1})^{-1/2} \begin{bmatrix} 0 & q^{1/2} \\ q^{-1/2} & 0 \end{bmatrix}, \quad (\overline{\sigma}_{0}^{-1})_{\dot{\alpha}\beta} = (q+q^{-1})^{-1/2} \begin{bmatrix} 0 & -q^{-1/2} \\ q^{1/2} & 0 \end{bmatrix}. \tag{8}$$

Definition 1.2. [8] We consider the set $U = \left\{ u_1^2, v^{\dot{1}2}, x_{1\dot{2}}^{\dot{1}}, z_{\dot{2}}^{\dot{1}}, y^{2\dot{1}}, t_{1\dot{2}} \right\} \subset C$. A function on the q - spinor variables is defined as $\Psi(U) = \Psi(u_1^2, v^{\dot{1}2}, x_{1\dot{2}}^{\dot{1}}, z_{\dot{2}}^{\dot{1}}, y^{2\dot{1}}, t_{1\dot{2}})$, where

$$\begin{aligned}
u_1^2 &\equiv \psi_1\psi^2 - q\psi^2\psi_1, \\
v^{\dot{1}2} &\equiv \psi^2\varphi^{\dot{1}} - q\varphi^{\dot{1}}\psi^2, \\
x_{1\dot{2}}^{\dot{1}} &\equiv \psi_1\varphi^{\dot{1}} - \varphi^{\dot{1}}\psi_1 + q(q+1)^{-1/2}\varphi_{\dot{2}}\varphi^{\dot{1}}, \\
y^{2\dot{1}} &\equiv \psi^2\varphi^{\dot{1}} - \varphi^{\dot{1}}\psi^2, \\
z_{\dot{2}}^{\dot{1}} &\equiv \psi_1\varphi_{\dot{2}} - q^{-1}\varphi_{\dot{2}}\varphi^{\dot{1}}, \\
t_{1\dot{2}} &\equiv \psi_1\varphi_{\dot{2}} - q\varphi_{\dot{2}}\psi_1.
\end{aligned}$$

Definition 1.3. [8] Let $f, g : U \longrightarrow C$ be functions and $u^\beta \in U$. The following properties are satisfy on the $q-$ spinor variables, we state some clear properties of the functions on the $q-$ spinor variables

1. $(f+g)(u^\beta) = f(u^\beta) + g(u^\beta)$.
2. $(f \cdot g)(u^\beta) = f(u^\beta) \cdot g(u^\beta)$.
3. $(f-g)(u^\beta) = f(u^\beta) - g(u^\beta)$.
4. $\left(\dfrac{f}{g}\right)(u^\beta) = \dfrac{f(u^\beta)}{g(u^\beta)}, \quad g(u^\beta) \neq 0.$

Definition 1.4. For a function $f : U \longrightarrow C$ and $u^\beta \in C$, the $q-$ spinor derivative is defined as [8] :

$$\frac{\mathrm{d}_q f}{\mathrm{d}_q u^\beta} = \frac{f((qu)^\beta) - qf(u^\beta)}{(qu)^\beta - qu^\beta}, \tag{9}$$

and its conjugate complex

$$\frac{\mathrm{d}_q f}{\mathrm{d}_q v^{\dot{\alpha}}} = \frac{f((qv)^{\dot{\alpha}}) - qf(v^{\dot{\alpha}})}{(qv)^{\dot{\alpha}} - qv^{\dot{\alpha}}}. \tag{10}$$

1.1. Clifford algebra and Dirac operator

Let $\{\gamma_1, \gamma_2, \cdots, \gamma_n,\}$ be an orthonormal basis of $\mathbb{R}^n$. The *Clifford algebra* is generated over $\mathbb{R}^n$ under the relation:

$$\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu = -2\delta_{\mu\nu}\gamma_0, \quad \gamma_\mu^2 = -|\gamma_\mu|^2\gamma_0, \quad \mu, \nu = 1, 2, ..., n, \tag{11}$$

where $\delta_{\mu\nu}$ is the Kronecker symbol (see [14], [3], [13] for more details). We will denote the Clifford algebra by Cl_n, and each element in Cl_n can be expressed by its components as $\sum_a \gamma_a x_a$, where $a = (\mu_1, ..., \mu_n)$ with each $\mu_l \in \{1, 2, ..., n\}$. Any element $\boldsymbol{x} \in \mathbb{R}^n$ can be identified with a 1-vector in the Clifford algebra [13]

$$(x_1, x_2, ..., x_n) \longrightarrow \boldsymbol{x} = x_1\gamma_1 + x_2\gamma_2 + \cdots + x_n\gamma_n. \tag{12}$$

On other hand, the Dirac operator used here is

$$D := \gamma_\mu \frac{\partial}{\partial x_\mu}, \tag{13}$$

we refer to reader to [3], [4], [18], [10] for more details.

1.2. $q-$ deformed Dirac matrices

Definition 1.5. The $q-$ deformed Dirac matrices are defined in [17], and are given by

$$\gamma_\mu := \begin{bmatrix} 0 & (\sigma_\mu)^\alpha_{\dot{\beta}} \\ (\overline{\sigma}_\mu)^{\dot{\alpha}}_\beta & 0 \end{bmatrix}, \tag{14}$$

where $(\sigma_\mu)^\alpha_{\dot{\beta}}$ and $(\overline{\sigma}_\mu)^{\dot{\alpha}}_\beta$ denote the Pauli matrices of $q-$ deformed Minkowski space (e.g. [16] for more details), and are defined as

$$\begin{aligned} (\sigma_+)^\alpha_{\dot{\beta}} &= \begin{bmatrix} 0 & 0 \\ 0 & kq^{1/2}\lambda_+^{1/2} \end{bmatrix}, \quad (\sigma_3)^\alpha_{\dot{\beta}} = k\begin{bmatrix} 0 & q \\ 1 & 0 \end{bmatrix}, \\ (\sigma_-)^\alpha_{\dot{\beta}} &= k\begin{bmatrix} q^{1/2}\lambda_+^{1/2} & 0 \\ 0 & 0 \end{bmatrix}, \quad (\sigma_0)^\alpha_{\dot{\beta}} = k\begin{bmatrix} 0 & -q^{-1} \\ 1 & 0 \end{bmatrix}, \end{aligned} \tag{15}$$

and their conjugated counterparts

$$\begin{aligned}(\overline{\sigma}_+)^{\alpha}_{\dot{\beta}} &= \begin{bmatrix}0 & 0\\ 0 & \overline{k}q^{-1/2}\lambda_+^{1/2}\end{bmatrix}, \quad (\overline{\sigma}_3)^{\alpha}_{\dot{\beta}} = \overline{k}\begin{bmatrix}0 & 1\\ q^{-1} & 0\end{bmatrix},\\ (\overline{\sigma}_-)^{\alpha}_{\dot{\beta}} &= \overline{k}\begin{bmatrix}q^{-1/2}\lambda_+^{1/2} & 0\\ 0 & 0\end{bmatrix}, \quad (\overline{\sigma}_0)^{\alpha}_{\dot{\beta}} = \overline{k}\begin{bmatrix}0 & 1\\ -q & 0\end{bmatrix},\end{aligned} \tag{16}$$

where $k, \overline{k}$ are characteristic parameters associated to bosons ($q = +1$) and fermions ($q = -1$).

1.3. Real spinors in the space

The real spinors in the space are defined based on the work of Zatloukal [19], which are 6 bivectors of the spacetime Clifford algebra

$$|\Psi\gamma_1\gamma_2\rangle = -i\hat{\gamma}_2|\Psi\rangle^*, \tag{17}$$
$$|\Psi\gamma_2\gamma_0\rangle = \hat{\gamma}_2|\Psi\rangle^*, \tag{18}$$
$$|\Psi\gamma_3\gamma_0\rangle = \hat{\gamma}_5|\Psi\rangle, \tag{19}$$
$$|\Psi\gamma_3\gamma_0\rangle = -\hat{\gamma}_2\hat{\gamma}_5|\Psi\rangle^*, \tag{20}$$
$$|\Psi\gamma_1\gamma_3\rangle = -i\hat{\gamma}_2\hat{\gamma}_5|\Psi\rangle^*, \tag{21}$$
$$|\Psi\gamma_2\gamma_1\rangle = i|\Psi\rangle, \tag{22}$$

where $|\Psi\rangle^* = (z_0^*, z_1^*, z_2^*, z_3^*)^T$, and we denoted $\hat{\gamma}_5 = i\hat{\gamma}_0\hat{\gamma}_1\hat{\gamma}_2\hat{\gamma}_3$ as common, and z_0^*, z_1^*, z_2^* and z_3^*

$$z_0^* = \langle\Psi^*(1 + i\gamma_2^*\gamma_1)\rangle, \tag{23}$$
$$z_1^* = \langle\gamma_1\gamma_3\Psi^*(1 + i\gamma_2^*\gamma_1)\rangle, \tag{24}$$
$$z_2^* = \langle\gamma_3\gamma_0\Psi^*(1 + i\gamma_2^*\gamma_1)\rangle, \tag{25}$$
$$z_3^* = \langle\gamma_1\gamma_2\Psi^*(1 + i\gamma_2^*\gamma_1)\rangle, \tag{26}$$

being $\hat{\gamma}$ the Dirac matrices in the standard representation respectively. Since $\gamma_0^2 = 1$ [19], [7], it follows readily that

$$\langle\gamma_\mu\gamma_\nu\Psi| = \langle\Psi|\hat{\gamma}_\mu\hat{\gamma}_\nu \quad \mu, \nu = 0, 1, 2, 3. \tag{27}$$

Motivation

Though the topic of this paper is q- real spinor calculus, the motivation comes from the study of q- Differential and integral calculus in spinor variables studied in [8] and the real spinors in the space based on the Zatloukal's work ([19]). According to the above, our interest here is to study and relate the q-differential and integral calculus on q-spinor variables with the real spinor in the space , and their implications with the differential equations. The main aim of this work is to study the q- differential and integral calculus and the differential equations on real spinor variables. Also, it is found the solutions to the differential equation in q-real spinor variables.

This paper builds on these developments by extending the spinor framework to real variables. While some preliminary definitions, such as those leading to Definition 1.3, directly reference prior work in [8], the current study introduces novel q-real differential operators, integral formulations, and differential equations. These contributions address specific cases where complex variables are unnecessary, providing a complementary perspective to the original work. By minimizing the repetition of results already presented in previous work and emphasizing new theoretical advancements, this study complements earlier efforts while highlighting its unique contributions. It provides a pathway to extend spinor calculus into real-variable domains, with potential applications in both mathematical theory and physical models. This paper is organized as follows. We briefly recall the preliminaries will be used in this paper in Sect.2. The q- differential operators for q-spinor variables, the q-spinor chain rule, the new q-differential operator, the q-Dirac differential operator, and the integral formulas in q- spinor variables are then proposed in Sect.3. In Sect.4 the differential equations in q-real spinor variables are stablished. In the Sect. 5, finally, some suggestions for further work are presented.

Notation

In the Section 2, we will denote by $\boldsymbol{x}_{\dot{\alpha}}^{\beta}$ instead of $\gamma_\mu\gamma_\nu u_{\dot{\alpha}}^{\beta}$, and the $q-$ real spinor derivative by $\frac{\boldsymbol{\partial}_q\psi}{\boldsymbol{\partial}_q\boldsymbol{x}_{\dot{\alpha}}^{\beta}}$.

2. q-Difference operators for q-real spinor variables

In this section we will mention about the q-Difference operators for q-real spinor variables considering the Definition1.4. To begin, first we define the function on q-real spinor

2.4. Functions on $q-$ real spinor variables

Proposition 2.1. *We consider the set* $u_{\dot{\alpha}}^{\beta} = \left\{u_1^2, v^{\dot{1}2}, x_{1\dot{2}}^{\dot{1}}, z_{\dot{2}}^{\dot{1}}, y^{2\dot{1}}, t_{1\dot{2}}\right\} \subset C$. *A function on the q - real spinor variables is defined as*

$$\psi(\gamma_\mu\gamma_\nu u_{\dot{\alpha}}^{\beta}) = \psi(\gamma_\mu\gamma_\nu u_1^2, \gamma_\mu\gamma_\nu v^{\dot{1}2}, \gamma_\mu\gamma_\nu x_{1\dot{2}}^{\dot{1}}, \gamma_\mu\gamma_\nu z_{\dot{2}}^{\dot{1}}, \gamma_\mu\gamma_\nu y^{2\dot{1}}, \gamma_\mu\gamma_\nu t_{1\dot{2}}). \tag{28}$$

Proof. It is sufficient to use (27) together with the observation that $\langle \gamma_\mu \gamma_\nu u^{\beta}_{\dot\alpha} | \psi \rangle = \psi(\gamma_\mu \gamma_\nu u^{\beta}_{\dot\alpha})$. ☑

Remark 2.2. For convenience we will to denote the function on $q-$ real spinor variables as $\psi(\boldsymbol{x}^{\beta}_{\dot\alpha})$.

Taking into account the above remark, we can define the following properties for functions on the q -real spinor variables similary to Definition1.3

Definition 2.3. Let $f, g : u^{\beta}_{\dot\alpha} \longrightarrow R^m$ be functions and $\boldsymbol{x}^{\alpha}_{\dot\beta} \in u^{\beta}_{\dot\alpha}$. The following properties are satisfy on the functions of q-real spinor variables, and we state some clear properties of the functions on the q-spinor variables

1. $(f + g)(\boldsymbol{x}^{\alpha}_{\dot\beta}) = f(\boldsymbol{x}^{\alpha}_{\dot\beta}) + g(\boldsymbol{x}^{\alpha}_{\dot\beta})$.
2. $(f \cdot g)(\boldsymbol{x}^{\alpha}_{\dot\beta}) = f(\boldsymbol{x}^{\alpha}_{\dot\beta}) \cdot g(\boldsymbol{x}^{\alpha}_{\dot\beta})$.
3. $(f - g)(\boldsymbol{x}^{\alpha}_{\dot\beta}) = f(\boldsymbol{x}^{\alpha}_{\dot\beta}) - g(\boldsymbol{x}^{\alpha}_{\dot\beta})$.
4. $\left(\dfrac{f}{g}\right)(\boldsymbol{x}^{\alpha}_{\dot\beta}) = \dfrac{f(\boldsymbol{x}^{\alpha}_{\dot\beta})}{g(\boldsymbol{x}^{\alpha}_{\dot\beta})}, \quad g(\boldsymbol{x}^{\alpha}_{\dot\beta}) \neq 0.$

2.5. q-Real spinor derivative

With the mathematical formalism of above section in hand, we are in a position to define the q-real spinor derivative which is mentioned in the following proposition

Proposition 2.4. *Let $\psi : u^{\beta}_{\dot\alpha} \longrightarrow R^m$. The q-real spinor derivative can be expressed by:*

$$\frac{\boldsymbol{\partial}_q \psi}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot\alpha}} = \frac{\psi(\boldsymbol{x}^{\beta}_{\dot\alpha} - q\boldsymbol{u}^{\beta}_{\dot\alpha}) + \psi(q\boldsymbol{u}^{\beta}_{\dot\alpha})}{\boldsymbol{x}^{\beta}_{\dot\alpha}}, \tag{29}$$

where $\boldsymbol{u}^{\beta}_{\dot\alpha} = (\gamma_\mu \gamma_\nu u)^{\beta}_{\dot\alpha}$.

Proof. From (9) we can see that

$$\frac{\boldsymbol{\partial}_q \psi}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot\alpha}} = \frac{\psi((q\gamma_\mu \gamma_\nu u)^{\beta}_{\dot\alpha}) - q\psi(\boldsymbol{u}^{\beta}_{\dot\alpha})}{(q\gamma_\mu \gamma_\nu u)^{\beta}_{\dot\alpha} - q\boldsymbol{u}^{\beta}_{\dot\alpha}}, \tag{30}$$

writing $(q\gamma_\mu \gamma_\nu u)^{\beta}_{\dot\alpha} = \boldsymbol{x}^{\beta}_{\dot\alpha} - q\boldsymbol{u}^{\beta}_{\dot\alpha}$ yields $(q\gamma_\mu \gamma_\nu u)^{\beta}_{\dot\alpha} - q\boldsymbol{u}^{\beta}_{\dot\alpha} = \boldsymbol{x}^{\beta}_{\dot\alpha}$, interchanging $q\psi(\boldsymbol{u}^{\beta}_{\dot\alpha})$ by $-\psi(q\boldsymbol{u}^{\beta}_{\dot\alpha})$, we can rewrite (30) as

$$\frac{\boldsymbol{\partial}_q \psi}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot\alpha}} = \frac{\psi(\boldsymbol{x}^{\beta}_{\dot\alpha} - q\boldsymbol{u}^{\beta}_{\dot\alpha}) + \psi(q\boldsymbol{u}^{\beta}_{\dot\alpha})}{\boldsymbol{x}^{\beta}_{\dot\alpha}},$$

this is the desired conclusion. ☑

Theorem 2.5. *Assume that* $\psi : u^{\beta}_{\dot{\alpha}} \longrightarrow R^m$ *and* $\varphi : u^{\beta}_{\dot{\alpha}} \longrightarrow R^m$ *are spinorial differentiable at* $\boldsymbol{x}^{\beta}_{\dot{\alpha}} \in R^m$. *Then*

1. *the sum* $\psi + \varphi : u^{\beta}_{\dot{\alpha}} \longrightarrow R^m$ *is q-differentiable at* $\boldsymbol{u}^{\beta}_{\dot{\alpha}}$ *and*

$$\frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}}(\psi + \varphi) = \frac{\boldsymbol{\partial}_q \psi}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}} + \frac{\boldsymbol{\partial}_q \varphi}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}}. \tag{31}$$

2. *the product* $\psi\varphi : u^{\beta}_{\dot{\alpha}} \longrightarrow R^m$ *is q-differentiable at* $\boldsymbol{u}^{\beta}_{\dot{\alpha}}$ *and*

$$\frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}}(\psi\varphi) = \frac{\boldsymbol{\partial}_q \psi}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}}\varphi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) - \psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}})\frac{\boldsymbol{\partial}_q \varphi}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}}. \tag{32}$$

3. $(\boldsymbol{x}^{\beta}_{\dot{\alpha}})^n : u^{\beta}_{\dot{\alpha}} \longrightarrow R^m$

$$\frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}}[(\boldsymbol{x}^{\beta}_{\dot{\alpha}})]^n = \sum_{k=1}^{n-1}(-1)^{k-1}(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}})^{n-k}(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})^{k-1}. \tag{33}$$

Proof. 1.

$$\begin{aligned}
\frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}}(\psi + \varphi) =& \frac{\psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \varphi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \varphi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})}{\boldsymbol{x}^{\beta}_{\dot{\alpha}}}, \\
=& \frac{\psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})}{\boldsymbol{x}^{\beta}_{\dot{\alpha}}} + \frac{\varphi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \varphi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})}{\boldsymbol{x}^{\beta}_{\dot{\alpha}}}, \\
=& \frac{\boldsymbol{\partial}_q \psi}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}} + \frac{\boldsymbol{\partial}_q \varphi}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}}.
\end{aligned}$$

2.

$$\frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}}(\psi\varphi) = \frac{\psi(\boldsymbol{x}^{\beta} - q\boldsymbol{u}^{\beta}_{\dot{\beta}})\varphi(\boldsymbol{x}^{\beta} - q\boldsymbol{u}^{\beta}_{\dot{\beta}}) + \psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})\varphi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})}{\boldsymbol{x}^{\beta}_{\dot{\alpha}}},$$

$$= \frac{\psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\beta}})\varphi(\boldsymbol{x}^{\beta} - q\boldsymbol{u}^{\beta}_{\dot{\beta}}) + \psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})\varphi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) - \psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})\varphi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})\varphi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})}{\boldsymbol{x}^{\beta}_{\dot{\alpha}}},$$

$$= \left[\frac{\psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})}{\boldsymbol{x}^{\beta}_{\dot{\alpha}}}\right]\varphi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}})\left[\frac{\varphi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) - \varphi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}})}{\boldsymbol{x}^{\beta}_{\dot{\alpha}}}\right],$$

for convenience we can interchange $\varphi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})$ by $-\varphi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})$ into the above expression, resulting

$$\left[\frac{\psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})}{\boldsymbol{x}^{\beta}_{\dot{\alpha}}}\right]\varphi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) - \psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}})\left[\frac{\varphi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \varphi(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}})}{\boldsymbol{x}^{\beta}_{\dot{\alpha}}}\right],$$

finally we obtain (32).

3.

$$\begin{aligned}\frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}}[(\boldsymbol{x}^{\beta}_{\dot{\alpha}})]^n &= \frac{(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\beta}})^n + (q\boldsymbol{u}^{\beta}_{\dot{\alpha}})^n}{\boldsymbol{x}^{\beta}_{\dot{\alpha}}}, \\ &= (\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\beta}})^{n-1} - (\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\beta}})^{n-2}(q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + ... + (q\boldsymbol{u}^{\beta}_{\dot{\alpha}}))^{n-1}, \\ &= \sum_{k=1}^{n-1}(-1)^{k-1}(\boldsymbol{x}^{\beta}_{\dot{\alpha}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}})^{n-k}(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})^{k-1}.\end{aligned}$$

☑

Example 2.6. *Let $\psi : u^{\beta}_{\dot{\alpha}} \longrightarrow R^m$ be a function on q-real spinor variables of the form $\psi(\boldsymbol{v}^{\mathrm{i}2}) = (\boldsymbol{v}^{\mathrm{i}2})^2 + q\boldsymbol{v}^{\mathrm{i}2}$. Applying (31) and (33) we have*

$$\frac{\boldsymbol{\partial}_q \psi}{\boldsymbol{\partial}_q \boldsymbol{v}^{\mathrm{i}2}} = (\boldsymbol{v}^{\mathrm{i}2} - q\boldsymbol{u}^{\mathrm{i}2})(\boldsymbol{v}^{\mathrm{i}2} - 2q\boldsymbol{u}^{\mathrm{i}2}) + q.$$

Remark 2.7. We assume that $\gamma_\mu \gamma_\nu u^{\beta}_{\dot{\alpha}} \neq (\gamma_\mu \gamma_\nu u)^{\beta}_{\dot{\alpha}}$.

This allows us to introduce the *q*- chain rule for real spinor variables.

2.6. q-chain rule for real spinor variables

Theorem 2.8. *[11] If Ψ and $\boldsymbol{x}^{\beta}_{\dot{\alpha}}$ are both differentiable at $x_j, j = 1,..,n$ and $\Psi(x_j)$ is the composite function defined by $\Psi[\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j)]$, then Ψ is differentiable and $\frac{\boldsymbol{\partial}_q \Psi}{\boldsymbol{\partial}_q x_j}$ is given by the product*

$$\frac{\boldsymbol{\partial}_q \Psi}{\boldsymbol{\partial}_q x_j} = \frac{\boldsymbol{\partial}_q \Psi}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}} \frac{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}}{\boldsymbol{\partial}_q x_j}. \tag{34}$$

Proof. The following assumptions will be needed throughout the proof:

$$\frac{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}}{\boldsymbol{\partial}_q x_j} = \frac{\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j - qx_j) + \boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j)}{x_j}, \tag{35}$$

$$\frac{\boldsymbol{\partial}_q \Psi}{\boldsymbol{\partial}_q x_j} = \frac{\Psi[\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j - qx_j) - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}(x_j - qx_j)] + \Psi[q\boldsymbol{u}^{\beta}_{\dot{\alpha}}(x_j)]}{x_j}. \tag{36}$$

According to the above assumptions, we can claim that (29) can be written as

$$\frac{\boldsymbol{\partial}_q \Psi}{\boldsymbol{\partial}_q \boldsymbol{x}^{\beta}_{\dot{\alpha}}} = \frac{\Psi[\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j - qx_j) - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}(x_j - qx_j)] + \Psi[q\boldsymbol{u}^{\beta}_{\dot{\alpha}}(x_j)]}{\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j - qx_j)}, \tag{37}$$

replacing the denominator of (37) by $\boldsymbol{x}_{\dot{\alpha}}^{\beta}(x_j - qx_j) + \boldsymbol{x}_{\dot{\alpha}}^{\beta}(x_j)$ yields

$$\frac{\boldsymbol{\partial}_q \Psi}{\boldsymbol{\partial}_q \boldsymbol{x}_{\dot{\alpha}}^{\beta}} = \frac{\Psi[\boldsymbol{x}_{\dot{\alpha}}^{\beta}(x_j - qx_j) - q\boldsymbol{u}_{\dot{\alpha}}^{\beta}(x_j - qx_j)] + \Psi[q\boldsymbol{u}_{\dot{\alpha}}^{\beta}(x_j)]}{\boldsymbol{x}_{\dot{\alpha}}^{\beta}(x_j - qx_j) + \boldsymbol{x}_{\dot{\alpha}}^{\beta}(x_j)}, \tag{38}$$

multipying both sides by $1/x_j$ we obtain

$$\frac{\boldsymbol{x}_{\dot{\alpha}}^{\beta}(x_j - qx_j) + \boldsymbol{x}_{\dot{\alpha}}^{\beta}(x_j)}{x_j} \frac{\boldsymbol{\partial}_q \Psi}{\boldsymbol{\partial}_q \boldsymbol{x}_{\dot{\alpha}}^{\beta}} = \frac{\Psi[\boldsymbol{x}_{\dot{\alpha}}^{\beta}(x_j - qx_j) - q\boldsymbol{u}_{\dot{\alpha}}^{\beta}(x_j - qx_j)] + \Psi[q\boldsymbol{u}_{\dot{\alpha}}^{\beta}(x_j)]}{x_j},$$

in virtue of (35) and (36), finally we get (34), and therefore the proof is complete. ☑

Remark 2.9. Similar considerations apply to $\boldsymbol{u}_{\dot{\alpha}}^{\beta}$, namely

$$\frac{\boldsymbol{\partial}_q \Psi}{\boldsymbol{\partial}_q x_j} = \frac{\boldsymbol{\partial}_q \Psi}{\boldsymbol{\partial}_q \boldsymbol{u}_{\dot{\alpha}}^{\beta}} \frac{\boldsymbol{\partial}_q \boldsymbol{u}_{\dot{\alpha}}^{\beta}}{\boldsymbol{\partial}_q x_j}, \tag{39}$$

and $\frac{\boldsymbol{\partial}_q \Psi}{\boldsymbol{\partial}_q \boldsymbol{u}_{\dot{\alpha}}^{\beta}}$ is given by

$$\frac{\boldsymbol{\partial}_q \Psi}{\boldsymbol{\partial}_q \boldsymbol{u}_{\dot{\alpha}}^{\beta}} = \frac{\Psi(\boldsymbol{u}_{\dot{\alpha}}^{\beta} - q\boldsymbol{x}_{\dot{\alpha}}^{\beta}) + \Psi(q\boldsymbol{x}_{\dot{\alpha}}^{\beta})}{\boldsymbol{u}_{\dot{\alpha}}^{\beta}}. \tag{40}$$

2.7. q-Difference operators for q-real spinor variables

Proposition 2.10. *The q-difference operator for q-real spinor variables can be expressed as*

$$\boldsymbol{D}_2^q = \hat{\gamma}_2 \frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q x_2}, \tag{41}$$

$$\boldsymbol{D}_j^q = i\hat{\gamma}_5 \frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q x_j}, \tag{42}$$

$$\underline{\boldsymbol{D}}_j^q = i\hat{\gamma}_2\hat{\gamma}_5 \frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q x_j}, \quad j = 1,..,5. \tag{43}$$

Proof. If we prove that the square of (41), (42), and (43) are equivalent to $-\frac{\boldsymbol{\partial}_q^2}{\boldsymbol{\partial}_q x_j^2}$ and $-\frac{\boldsymbol{\partial}_q^2}{\boldsymbol{\partial}_q x_2^2}$, then the assertion follows. ☑

Remark 2.11. The expression (41) is called the *the q- Dirac real operator*.

Definition 2.12. From the (23), (24), (25), and (26), we define the q- conjugated real spinor variables as

$$\boldsymbol{v}_{\dot{0}} = \langle \boldsymbol{x}_{\dot{\alpha}}^{\beta}(1 + i\gamma_2^*\gamma_1)\rangle, \quad \boldsymbol{p}_{\dot{0}} = \langle \boldsymbol{u}_{\dot{\alpha}}^{\beta}(1 + i\gamma_2^*\gamma_1)\rangle \tag{44}$$

$$\boldsymbol{v}_{\dot{1}} = \langle \gamma_1\gamma_3\boldsymbol{x}_{\dot{\alpha}}^{\beta}(1 + i\gamma_2^*\gamma_1)\rangle, \quad \boldsymbol{p}_{\dot{1}} = \langle \gamma_1\gamma_3\boldsymbol{u}_{\dot{\alpha}}^{\beta}(1 + i\gamma_2^*\gamma_1)\rangle \tag{45}$$

$$\boldsymbol{v}_{\dot{2}} = \langle \gamma_3\gamma_0\boldsymbol{x}_{\dot{\alpha}}^{\beta}(1 + i\gamma_2^*\gamma_1)\rangle, \quad \boldsymbol{p}_{\dot{2}} = \langle \gamma_3\gamma_0\boldsymbol{u}_{\dot{\alpha}}^{\beta}(1 + i\gamma_2^*\gamma_1)\rangle \tag{46}$$

$$\boldsymbol{v}_{\dot{3}} = \langle \gamma_1\gamma_2\boldsymbol{x}_{\dot{\alpha}}^{\beta}(1 + i\gamma_2^*\gamma_1)\rangle, \quad \boldsymbol{p}_{\dot{3}} = \langle \gamma_1\gamma_2\boldsymbol{u}_{\dot{\alpha}}^{\beta}(1 + i\gamma_2^*\gamma_1)\rangle. \tag{47}$$

From the above definition we can construct a function on the q- conjugated real spinor variables of the following form: $\psi : (\boldsymbol{u}_{\dot{k}}, \boldsymbol{p}_{\dot{k}}) \longrightarrow R^m$ for all $0 \leq \dot{k} \leq 3$, this is $\psi(\boldsymbol{v}_{\dot{k}}, \boldsymbol{p}_{\dot{k}})$.

Theorem 2.13. *For a function $\psi : (\boldsymbol{v}_{\dot{k}}, \boldsymbol{p}_{\dot{k}}) \longrightarrow R^m$, the q-conjugated derivatives are defined as*

$$\frac{\boldsymbol{\partial}_q \psi}{\boldsymbol{\partial}_q \boldsymbol{v}_{\dot{k}}} = \frac{\psi(\boldsymbol{v}_{\dot{k}} - q\boldsymbol{x}^{\beta}_{\dot{\alpha}}) + \psi(q\boldsymbol{x}^{\beta}_{\dot{\alpha}})}{\boldsymbol{v}_{\dot{k}}}, \tag{48}$$

$$\frac{\boldsymbol{\partial}_q \psi}{\boldsymbol{\partial}_q \boldsymbol{p}_{\dot{k}}} = \frac{\psi(\boldsymbol{p}_{\dot{k}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})}{\boldsymbol{p}_{\dot{k}}}. \tag{49}$$

Proof. It suffices to make the substitution $\boldsymbol{x}^{\beta}_{\dot{\alpha}}$ by $\boldsymbol{v}_{\dot{k}}$ and $\boldsymbol{p}_{\dot{k}}$ into (30), which the proof is complete ☑

Theorem 2.14. *The q-difference operators associated to conjugated real spinor variables are given by*

$$\boldsymbol{D}_q = \frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q \boldsymbol{v}_{\dot{0}}} + \gamma_1\gamma_3 \frac{\boldsymbol{\partial_q}}{\boldsymbol{\partial}_q \boldsymbol{v}_{\dot{1}}} + i\gamma_3\gamma_{\dot{0}} \frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q \boldsymbol{v}_{\dot{2}}} + \gamma_1\gamma_2 \frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q \boldsymbol{v}_{\dot{3}}}, \tag{50}$$

$$\boldsymbol{D}'_q = \frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q \boldsymbol{p}_{\dot{0}}} + \gamma_1\gamma_3 \frac{\boldsymbol{\partial_q}}{\boldsymbol{\partial}_q \boldsymbol{p}_{\dot{1}}} + i\gamma_3\gamma_0 \frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q \boldsymbol{p}_{\dot{2}}} + \gamma_1\gamma_2 \frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q \boldsymbol{p}_{\dot{3}}}. \tag{51}$$

Proof. Analysis similar to that in the proof of Proposition 2.7 shows that $\boldsymbol{D}^2_q = \frac{\boldsymbol{\partial}^2_q}{\boldsymbol{\partial}_q \boldsymbol{v}^2_{\dot{0}}} - \frac{\boldsymbol{\partial}^2_q}{\boldsymbol{\partial}_q \boldsymbol{v}^2_{\dot{1}}} - \frac{\boldsymbol{\partial}^2_q}{\boldsymbol{\partial}_q \boldsymbol{v}^2_{\dot{2}}} - \frac{\boldsymbol{\partial}^2_q}{\boldsymbol{\partial}_q \boldsymbol{v}^2_{\dot{3}}}$ and $\boldsymbol{D}'^2_q = \frac{\boldsymbol{\partial}^2_q}{\boldsymbol{\partial}_q \boldsymbol{p}^2_{\dot{0}}} - \frac{\boldsymbol{\partial}^2_q}{\boldsymbol{\partial}_q \boldsymbol{p}^2_{\dot{1}}} - \frac{\boldsymbol{\partial}^2_q}{\boldsymbol{\partial}_q \boldsymbol{p}^2_{\dot{2}}} - \frac{\boldsymbol{\partial}^2_q}{\boldsymbol{\partial}_q \boldsymbol{p}^2_{\dot{3}}}$. ☑

Remark 2.15. The q-difference operators (50) and (51) can be written in terms of (48) and (49) in explicit form as

$$\boldsymbol{D}^{\dot{1}}_q \psi = \gamma_1\gamma_3 \left[\frac{\psi(\boldsymbol{v}_{\dot{1}} - q\boldsymbol{x}^{\beta}_{\dot{\alpha}}) + \psi(q\boldsymbol{x}^{\beta}_{\dot{\alpha}})}{\boldsymbol{v}_{\dot{1}}}\right], \quad \boldsymbol{D}^{\dot{2}}_q \psi = i\gamma_3\gamma_0 \left[\frac{\psi(\boldsymbol{v}_{\dot{2}} - q\boldsymbol{x}^{\beta}_{\dot{\alpha}}) + \psi(q\boldsymbol{x}^{\beta}_{\dot{\alpha}})}{\boldsymbol{v}_{\dot{2}}}\right], \tag{52}$$

$$\boldsymbol{D}^{\dot{3}}_q \psi = \gamma_1\gamma_2 \left[\frac{\psi(\boldsymbol{v}_{\dot{3}} - q\boldsymbol{x}^{\beta}_{\dot{\alpha}}) + \psi(q\boldsymbol{x}^{\beta}_{\dot{\alpha}})}{\boldsymbol{v}_{\dot{3}}}\right], \quad \boldsymbol{D}^{\dot{0}}_q \psi = \frac{\psi(\boldsymbol{v}_{\dot{0}} - q\boldsymbol{x}^{\beta}_{\dot{\alpha}}) + \psi(q\boldsymbol{x}^{\beta}_{\dot{\alpha}})}{\boldsymbol{v}_{\dot{0}}}, \tag{53}$$

$$\boldsymbol{D}^{\dot{1}}_q \psi = \gamma_1\gamma_3 \left[\frac{\psi(\boldsymbol{p}_{\dot{1}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})}{\boldsymbol{p}_{\dot{1}}}\right], \quad \boldsymbol{D}^{\dot{2}}_q \psi = i\gamma_3\gamma_0 \left[\frac{\psi(\boldsymbol{p}_{\dot{2}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})}{\boldsymbol{p}_{\dot{2}}}\right], \tag{54}$$

$$\boldsymbol{D}^{\dot{3}}_q \psi = \gamma_1\gamma_2 \left[\frac{\psi(\boldsymbol{p}_{\dot{3}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})}{\boldsymbol{p}_{\dot{3}}}\right], \quad \boldsymbol{D}^{\dot{0}}_q \psi = \frac{\psi(\boldsymbol{p}_{\dot{0}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})}{\boldsymbol{p}_{\dot{0}}}. \tag{55}$$

With these results, it is possible to define the q-spinor real integral formulas by the following theorem.

3. q-Spinor real integral formulas

Theorem 3.1. *Let $\psi : u^{\beta}_{\dot{\alpha}} \longrightarrow R^m$ and let Ω_q be a subset over a manifold $\mathcal{M}$ in R^m, the q-spinor real integral formulas of the q- spinor conjugated variables are given by*

$$\int_{\Omega_q} \frac{\psi(q\boldsymbol{v}_{\dot{k}})\mathbf{d}_q\boldsymbol{v}_{\dot{k}}}{\boldsymbol{v}_{\dot{k}} + \boldsymbol{x}^{\beta}_{\dot{\alpha}}} = q\sum_{n=0}^{\infty}[\gamma^{\mu}\gamma^{\nu}\psi(q\boldsymbol{x}^{\beta}_{\dot{\alpha}})]^n, \tag{56}$$

$$\int_{\Omega_q} \frac{\psi[(1-q)\boldsymbol{v}_{\dot{k}}]\mathbf{d}_q\boldsymbol{v}_{\dot{k}}}{\boldsymbol{v}_{\dot{k}} + \boldsymbol{x}^{\beta}_{\dot{\alpha}}} = q\sum_{n=0}^{\infty}[\gamma^{\mu}\gamma^{\nu}\psi[(1-q)\boldsymbol{x}^{\beta}_{\dot{\alpha}}]]^n, \tag{57}$$

$$\int_{\Omega_q} \frac{\psi(q\boldsymbol{p}_{\dot{k}})\mathbf{d}_q\boldsymbol{p}_{\dot{k}}}{\boldsymbol{p}_{\dot{k}} + \boldsymbol{u}^{\beta}_{\dot{\alpha}}} = q\sum_{n=0}^{\infty}[\gamma^{\mu}\gamma^{\nu}\psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})]^n, \tag{58}$$

$$\int_{\Omega_q} \frac{\psi[(1-q)\boldsymbol{p}_{\dot{k}}]\mathbf{d}_q\boldsymbol{p}_{\dot{k}}}{\boldsymbol{p}_{\dot{k}} + \boldsymbol{u}^{\beta}_{\dot{\alpha}}} = q\sum_{n=0}^{\infty}[\gamma^{\mu}\gamma^{\nu}\psi[(1-q)\boldsymbol{u}^{\beta}_{\dot{\alpha}}]]^n. \tag{59}$$

Proof. First, we present the following changes of variables in (48), $q\boldsymbol{v}_{\dot{k}} = \boldsymbol{v}_{\dot{k}} - q\boldsymbol{x}^{\beta}_{\dot{\alpha}}$ and $(q^{-1}-1)\boldsymbol{v}_{\dot{k}} = \boldsymbol{x}^{\beta}_{\dot{\alpha}}$, obtaining

$$\frac{\boldsymbol{\partial}_q\psi}{\boldsymbol{\partial}_q\boldsymbol{v}_{\dot{k}}} = \frac{\psi(q\boldsymbol{v}_{\dot{k}}) + \psi[q(q^{-1}-1)\boldsymbol{v}_{\dot{k}}]}{q\boldsymbol{v}_{\dot{k}} + q\boldsymbol{x}^{\beta}_{\dot{\alpha}}}, \tag{60}$$

multiplyng both sides by $\mathbf{d}_q\boldsymbol{v}_{\dot{k}}$ we get

$$\mathbf{d}_q\psi = \frac{\psi(q\boldsymbol{v}_{\dot{k}})\mathbf{d}_q\boldsymbol{v}_{\dot{k}}}{q\boldsymbol{v}_{\dot{k}} + q\boldsymbol{x}^{\beta}_{\dot{\alpha}}} + \frac{\psi[(1-q)\boldsymbol{v}_{\dot{k}}]\mathbf{d}_q\boldsymbol{v}_{\dot{k}}}{q\boldsymbol{v}_{\dot{k}} + q\boldsymbol{x}^{\beta}_{\dot{\alpha}}}, \tag{61}$$

integrating both sides over Ω_q

$$\int_{\Omega_q}\mathbf{d}_q\psi = \int_{\Omega_q}\frac{\psi(q\boldsymbol{v}_{\dot{k}})\mathbf{d}_q\boldsymbol{v}_{\dot{k}}}{q\boldsymbol{v}_{\dot{k}} + q\boldsymbol{x}^{\beta}_{\dot{\alpha}}} + \int_{\Omega_q}\frac{\psi[(1-q)\boldsymbol{v}_{\dot{k}}]\mathbf{d}_q\boldsymbol{v}_{\dot{k}}}{q\boldsymbol{v}_{\dot{k}} + q\boldsymbol{x}^{\beta}_{\dot{\alpha}}}, \tag{62}$$

hence, to solve the integral $\int_{\Omega_q}\mathbf{d}_q\psi$, we will use similary the proof of the Theorem 2.9 of the reference [8] , to obtain

$$\sum_{n=0}^{\infty}[\gamma^{\mu}\gamma^{\nu}\psi(q\boldsymbol{x}^{\beta}_{\dot{\alpha}})]^n + \sum_{n=0}^{\infty}[\gamma^{\mu}\gamma^{\nu}\psi[(\boldsymbol{x}^{\beta}_{\dot{\alpha}})(1-q)]]^n,$$
$$= \int_{\Omega_q}\frac{\psi(q\boldsymbol{v}_{\dot{k}})\mathbf{d}_q\boldsymbol{v}_{\dot{k}}}{q\boldsymbol{v}_{\dot{k}} + q\boldsymbol{x}^{\beta}_{\dot{\alpha}}} + \int_{\Omega_q}\frac{\psi[(1-q)\boldsymbol{v}_{\dot{k}}]\mathbf{d}_q\boldsymbol{v}_{\dot{k}}}{q\boldsymbol{v}_{\dot{k}} + q\boldsymbol{x}^{\beta}_{\dot{\alpha}}},$$

finally we get

$$\int_{\Omega_q} \frac{\psi(q\boldsymbol{v}_{\dot{k}})\mathbf{d}_q\boldsymbol{v}_{\dot{k}}}{q\boldsymbol{v}_{\dot{k}}+q\boldsymbol{x}^{\beta}_{\dot{\alpha}}} = \sum_{n=0}^{\infty}[\gamma^{\mu}\gamma^{\nu}\psi(q\boldsymbol{x}^{\beta}_{\dot{\alpha}})]^n, \tag{63}$$

$$\int_{\Omega_q} \frac{\psi[(1-q)\boldsymbol{v}_{\dot{k}}]\mathbf{d}_q\boldsymbol{v}_{\dot{k}}}{q\boldsymbol{v}_{\dot{k}}+q\boldsymbol{x}^{\beta}_{\dot{\alpha}}} = \sum_{n=0}^{\infty}[\gamma^{\mu}\gamma^{\nu}\psi[(1-q)(\boldsymbol{x}^{\beta}_{\dot{\alpha}})]]^n. \tag{64}$$

The same process is applied to get (58) and (59) from (49) using the changes of variables $q\boldsymbol{p}_{\dot{k}} = \boldsymbol{p}_{\dot{k}} - q\boldsymbol{u}^{\beta}_{\dot{\alpha}}$ and $\boldsymbol{u}^{\beta}_{\dot{\alpha}} = \boldsymbol{p}_{\dot{k}}(q^{-1}-1)$. ☑

On other hand, we can stablish the q-spinor real integral from (40), (41), (42), and (43) by our next theorem.

Theorem 3.2. *The q-spinor real integral formulas over Ω_q associated to q-difference operators (41), (42), and (43) are given by*

$$\int_{\Omega_q} \frac{\psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})\mathbf{d}_q\boldsymbol{u}^{\beta}_{\dot{\alpha}}}{q\boldsymbol{u}^{\beta}_{\dot{\alpha}}+\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_2)} = \sum_{m=0}^{\infty}[\gamma^2\psi(q\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_2))]^m, \tag{65}$$

$$\int_{\Omega_q} \frac{\psi(q\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_2))\mathbf{d}_q\boldsymbol{u}^{\beta}_{\dot{\alpha}}}{q\boldsymbol{u}^{\beta}_{\dot{\alpha}}+\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_2)} = \sum_{m=0}^{\infty}[\gamma^2\psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_2))]^m, \tag{66}$$

$$\int_{\Omega_q} \frac{\psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})\mathbf{d}_q\boldsymbol{u}^{\beta}_{\dot{\alpha}}}{q\boldsymbol{u}^{\beta}_{\dot{\alpha}}+\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j)} = \sum_{m=0}^{\infty}[i\gamma^5\psi(q\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j))]^m, \tag{67}$$

$$\int_{\Omega_q} \frac{\psi(q\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j))\mathbf{d}_q\boldsymbol{u}^{\beta}_{\dot{\alpha}}}{q\boldsymbol{u}^{\beta}_{\dot{\alpha}}+\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j)} = \sum_{m=0}^{\infty}[i\gamma^5\psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_2))]^m, \tag{68}$$

$$\int_{\Omega_q} \frac{\psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})\mathbf{d}_q\boldsymbol{u}^{\beta}_{\dot{\alpha}}}{q\boldsymbol{u}^{\beta}_{\dot{\alpha}}+\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j)} = \sum_{m=0}^{\infty}[i\gamma^2\gamma^5\psi(q\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j))]^m, \tag{69}$$

$$\int_{\Omega_q} \frac{\psi(q\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j))\mathbf{d}_q\boldsymbol{u}^{\beta}_{\dot{\alpha}}}{q\boldsymbol{u}^{\beta}_{\dot{\alpha}}+\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j)} = \sum_{m=0}^{\infty}[i\gamma^2\gamma^5\psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j))]^m. \tag{70}$$

Proof. For the operator (41), first we rewrite (40) replacing $\boldsymbol{u}^{\beta}_{\dot{\alpha}} - \boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_2)$ by $q\boldsymbol{u}^{\beta}_{\dot{\alpha}}$ to obtain

$$\begin{aligned}\frac{\boldsymbol{\partial}_q\psi}{\boldsymbol{\partial}_q\boldsymbol{u}^{\beta}_{\dot{\alpha}}} &= \frac{\psi[\boldsymbol{u}^{\beta}_{\dot{\alpha}} - q\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_2)] + \psi(q\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_2))}{\boldsymbol{u}^{\beta}_{\dot{\alpha}}}, \\ &= \frac{\psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \psi(q\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_2))}{q\boldsymbol{u}^{\beta}_{\dot{\alpha}}+\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_2)},\end{aligned}$$

substituting the above expression into (39)

$$\frac{\boldsymbol{\partial}_q \Psi}{\boldsymbol{\partial}_q x_2} = \left[\frac{\psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}}) + \psi(q\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_2))}{q\boldsymbol{u}^{\beta}_{\dot{\alpha}} + \boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_2)} \right] \frac{\boldsymbol{\partial}_q \boldsymbol{u}^{\beta}_{\dot{\alpha}}}{\boldsymbol{\partial}_q x_2}, \tag{71}$$

multiplying both sides by $\mathbf{d}_q x_2$ we get

$$\mathbf{d}_q \Psi = \frac{\psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})\mathbf{d}_q \boldsymbol{u}^{\beta}_{\dot{\alpha}}}{q\boldsymbol{u}^{\beta}_{\dot{\alpha}} + \boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_2)} + \frac{\psi(q\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_2))\mathbf{d}_q \boldsymbol{u}^{\beta}_{\dot{\alpha}}}{q\boldsymbol{u}^{\beta}_{\dot{\alpha}} + \boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_2)}, \tag{72}$$

integrating both sides over Ω_q

$$\int_{\Omega_q} \mathbf{d}_q \Psi = \int_{\Omega_q} \frac{\psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})\mathbf{d}_q \boldsymbol{u}^{\beta}_{\dot{\alpha}}}{q\boldsymbol{u}^{\beta}_{\dot{\alpha}} + \boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_2)} + \int_{\Omega_q} \frac{\psi(q\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_2))\mathbf{d}_q \boldsymbol{u}^{\beta}_{\dot{\alpha}}}{q\boldsymbol{u}^{\beta}_{\dot{\alpha}} + \boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_2)}, \tag{73}$$

we can now proceed analogously to the proof of above theorem, obtaining (65) and (66)

$$\begin{aligned} \int_{\Omega_q} \frac{\psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})\mathbf{d}_q \boldsymbol{u}^{\beta}_{\dot{\alpha}}}{q\boldsymbol{u}^{\beta}_{\dot{\alpha}} + \boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_2)} + \int_{\Omega_q} \frac{\psi(q\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j))\mathbf{d}_q \boldsymbol{u}^{\beta}_{\dot{\alpha}}}{q\boldsymbol{u}^{\beta}_{\dot{\alpha}} + \boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_2)} \\ = \sum_{m=0}^{\infty} \left[\gamma^2 \psi(q\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_2))\right]^m + \sum_{m=0}^{\infty} \left[\gamma^2 \psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_2))\right]^m, \end{aligned} \tag{74}$$

therefore

$$\begin{aligned} \int_{\Omega_q} \frac{\psi(q\boldsymbol{u}^{\beta}_{\dot{\alpha}})\mathbf{d}_q \boldsymbol{u}^{\beta}_{\dot{\alpha}}}{q\boldsymbol{u}^{\beta}_{\dot{\alpha}} + \boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_2)} &= \sum_{m=0}^{\infty} \left[\gamma^2 \psi(q\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_2))\right]^m, \\ \int_{\Omega_q} \frac{\psi(q\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_j))\mathbf{d}_q \boldsymbol{u}^{\beta}_{\dot{\alpha}}}{q\boldsymbol{u}^{\beta}_{\dot{\alpha}} + \boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_2)} &= \sum_{m=0}^{\infty} \left[\gamma^2 \psi(\boldsymbol{x}^{\beta}_{\dot{\alpha}}(x_2))\right]^m. \end{aligned}$$

The same reasoning applies to the operators (42) and (43), to obtain (67), (68), (69), and (70). ☑

Let us mention an important consequence of the above theorem.

4. Differential equations in q-real spinor variables

Let us consider the following q- real spinor differential equation

$$(\boldsymbol{D}_q^{\mathrm{i}} - m)\psi(\boldsymbol{v}_{\mathrm{i}}) = 0, \quad m \in R. \tag{75}$$

In order to get solution of (75), it is necessary to put the following condition on ψ

$$\int_{\Omega_q} \boldsymbol{D}_q^{\mathrm{i}} \psi \mathbf{d}_q \boldsymbol{v}_{\mathrm{i}} = \psi(\boldsymbol{x}_{\dot{\alpha}}^{\beta}), \quad \alpha, \dot{\beta} = 1, 2. \tag{76}$$

and the following lemma

Lemma 4.1. *The integral (76) can be expressed in virtue of (56) as*

$$\int_{\Omega_q} \boldsymbol{D}_q^{\mathrm{i}} \psi \mathbf{d}_q \boldsymbol{v}_{\mathrm{i}} = \gamma_1\gamma_3 \left[\sum_{n=0}^{\infty} [\gamma^1\gamma^3 \psi(q\boldsymbol{x}_{\dot{\alpha}}^{\beta})]^n + \sum_{n=0}^{\infty} [\gamma^1\gamma^3 \psi((1-q)\boldsymbol{x}_{\dot{\alpha}}^{\beta})]^n \right]. \tag{77}$$

Proof. According to (52), the operator $\boldsymbol{D}_q^{\mathrm{i}}$ is defined as $\gamma_1\gamma_3 \frac{\boldsymbol{\partial}_q}{\boldsymbol{\partial}_q \boldsymbol{v}_{\mathrm{i}}}$. Therefore (61) can be expressed as

$$\boldsymbol{D}_q^{\mathrm{i}} \psi \mathbf{d}_q \boldsymbol{v}_{\mathrm{i}} = \gamma_1\gamma_3 \left[\frac{\psi(q\boldsymbol{v}_{\mathrm{i}})\mathbf{d}_q \boldsymbol{v}_{\mathrm{i}}}{q\boldsymbol{v}_{\mathrm{i}} + q\boldsymbol{x}_{\dot{\alpha}}^{\beta}} + \frac{\psi[(1-q)\boldsymbol{v}_{\mathrm{i}}]\mathbf{d}_q \boldsymbol{v}_{\mathrm{i}}}{q\boldsymbol{v}_{\mathrm{i}} + q\boldsymbol{x}_{\dot{\alpha}}^{\beta}} \right], \tag{78}$$

integrating both sides over Ω_q,

$$\int_{\Omega_q} \boldsymbol{D}_q^{\mathrm{i}} \psi \mathbf{d}_q \boldsymbol{v}_{\mathrm{i}} = \gamma_1\gamma_3 \left[\int_{\Omega_q} \frac{\psi(q\boldsymbol{v}_{\mathrm{i}})\mathbf{d}_q \boldsymbol{v}_{\mathrm{i}}}{q\boldsymbol{v}_{\mathrm{i}} + q\boldsymbol{x}_{\dot{\alpha}}^{\beta}} + \int_{\Omega_q} \frac{\psi[(1-q)\boldsymbol{v}_{\mathrm{i}}]\mathbf{d}_q \boldsymbol{v}_{\mathrm{i}}}{q\boldsymbol{v}_{\mathrm{i}} + q\boldsymbol{x}_{\dot{\alpha}}^{\beta}} \right], \tag{79}$$

the right side of (79) may be equalated to (63) and (64) finally we obtain

$$\int_{\Omega_q} \boldsymbol{D}_q^{\mathrm{i}} \psi \mathbf{d}_q \boldsymbol{v}_{\mathrm{i}} = \gamma_1\gamma_3 \left[\sum_{n=0}^{\infty} [\gamma^1\gamma^3 \psi(q\boldsymbol{x}_{\dot{\alpha}}^{\beta})]^n + \sum_{n=0}^{\infty} [\gamma^1\gamma^3 \psi((1-q)\boldsymbol{x}_{\dot{\alpha}}^{\beta})]^n \right],$$

which is our claim. ☑

Remark 4.2. The expressions (76) and (77) are equivalent.

The solution of (75) is established by our next theorem.

Theorem 4.3. *The solution of (75) over the subset Ω_q can be written as*

$$\int_{\Omega_q} \psi(\boldsymbol{v}_{\mathrm{i}})\mathbf{d}_q\boldsymbol{v}_{\mathrm{i}} = \frac{1}{m}\gamma_1\gamma_3\left[\sum_{n=0}^{\infty}[\gamma^1\gamma^3\psi(q\boldsymbol{x}^{\beta}_{\dot\alpha})]^n + \sum_{n=0}^{\infty}[\gamma^1\gamma^3\psi((1-q)\boldsymbol{x}^{\beta}_{\dot\alpha})]^n\right]. \tag{80}$$

Proof. We begin by rewriting (75) as

$$\boldsymbol{D}^{\mathrm{i}}_q\psi(\boldsymbol{v}_{\mathrm{i}}) = m\psi(\boldsymbol{v}_{\mathrm{i}}), \tag{81}$$

multiplying by $\mathbf{d}_q\boldsymbol{v}_{\mathrm{i}}$,

$$\boldsymbol{D}^{\mathrm{i}}_q\psi(\boldsymbol{v}_{\mathrm{i}})\mathbf{d}_q\boldsymbol{v}_{\mathrm{i}} = m\psi(\boldsymbol{v}_{\mathrm{i}})\mathbf{d}_q\boldsymbol{v}_{\mathrm{i}}, \tag{82}$$

integrating both sides over the subset Ω_q and using (77)

$$\begin{aligned} \int_{\Omega_q}\boldsymbol{D}^{\mathrm{i}}_q\psi(\boldsymbol{v}_{\mathrm{i}})\mathbf{d}_q\boldsymbol{v}_{\mathrm{i}} &= m\int_{\Omega_q}\psi(\boldsymbol{v}_{\mathrm{i}})\mathbf{d}_q\boldsymbol{v}_{\mathrm{i}}, \\ \gamma_1\gamma_3\left[\sum_{n=0}^{\infty}[\gamma^1\gamma^3\psi(q\boldsymbol{x}^{\beta}_{\dot\alpha})]^n + \sum_{n=0}^{\infty}[\gamma^1\gamma^3\psi((1-q)\boldsymbol{x}^{\beta}_{\dot\alpha})]^n\right] &= m\int_{\Omega_q}\psi(\boldsymbol{v}_{\mathrm{i}})\mathbf{d}_q\boldsymbol{v}_{\mathrm{i}}, \end{aligned}$$

finally we get (80). ☑

However, the solution of (75) is not unique. According to (76) we can say that the solution of (75) is given by

$$\gamma_1\gamma_3\psi(\boldsymbol{x}^{\beta}_{\dot\alpha}) = m\int_{\Omega_q}\psi(\boldsymbol{v}_{\mathrm{i}})\mathbf{d}_q\boldsymbol{v}_{\mathrm{i}}, \tag{83}$$

$$\psi(\boldsymbol{x}^{\beta}_{\dot\alpha}) = \left[\sum_{n=0}^{\infty}[\gamma^1\gamma^3\psi(q\boldsymbol{x}^{\beta}_{\dot\alpha})]^n + \sum_{n=0}^{\infty}[\gamma^1\gamma^3\psi((1-q)\boldsymbol{x}^{\beta}_{\dot\alpha})]^n\right]. \tag{84}$$

Let us to consider the following examples

Example 4.4. *Let $\boldsymbol{D}^{\dot 2}_q\psi(\boldsymbol{p}_{\dot 2}) = 0$ be a Differential equation in q-real spinor variable. This is a trivial case, where the solution is $\mathbf{0}$. For this case, is said that the solution is monogenic.*

Example 4.5. *For $a\boldsymbol{D}_q^{\dot{3}}\psi(\boldsymbol{v}_{\dot{3}}) + bf(\boldsymbol{v}_{\dot{3}}) = 0$ for all $a, b \in R$. Using the results previously mentioned, the solution is given by*

$$\psi(\boldsymbol{x}_{\dot{\alpha}}^{\beta}) = -\frac{b}{a}\left[\sum_{n=0}^{\infty}[\gamma^1\gamma^2 f(q\boldsymbol{x}_{\dot{\alpha}}^{\beta})]^n + \sum_{n=0}^{\infty}[\gamma^1\gamma^2 f((1-q)\boldsymbol{x}_{\dot{\alpha}}^{\beta})]^n\right].$$

Example 4.6. *Let us consider the following differential equation*

$$(\boldsymbol{D}_j^q - \lambda)\psi(\boldsymbol{x}_{\dot{\alpha}}^{\beta}(x_j)) = 0, \quad \lambda \in R. \tag{85}$$

In order to solve (85), it is sufficient to consider the following condition taking into account (42)

$$\int_{\Omega_q} \boldsymbol{D}_j^q \psi(\boldsymbol{x}_{\dot{\alpha}}^{\beta}(x_j))\mathbf{d}_q\boldsymbol{x}_{\dot{\alpha}}^{\beta} = i\hat{\gamma}_5\psi(x_j), \tag{86}$$

integrating both sides over Ω_q

$$\begin{aligned}\int_{\Omega_q} \boldsymbol{D}_j^q \psi(\boldsymbol{x}_{\dot{\alpha}}^{\beta}(x_j))\mathbf{d}_q\boldsymbol{x}_{\dot{\alpha}}^{\beta} =&\ \lambda\int_{\Omega_q}\psi(\boldsymbol{x}_{\dot{\alpha}}^{\beta}(x_j))\mathbf{d}_q\boldsymbol{x}_{\dot{\alpha}}^{\beta} \\ i\hat{\gamma}_5\psi(x_j) =&\ i\hat{\gamma}_5\lambda\left\{\sum_{m=0}^{\infty}[i\hat{\gamma}^5\psi(q(x_j)]^m + \sum_{m=0}^{\infty}[i\hat{\gamma}^5\psi((1-q)x_j)]^m\right\}.\end{aligned}$$

therefore

$$\psi(x_j) = \lambda\left\{\sum_{m=0}^{\infty}[i\hat{\gamma}^5\psi(q(x_j))]^m + \sum_{m=0}^{\infty}[i\hat{\gamma}^5\psi((1-q)x_j)]^m\right\}.$$

In the following example, we propose a solution to the q-**Dirac-Hestenes equation**, based on the non-deformed version described by Zatloukal and Hestenes [19], [6].

Proposition 4.7. *Let us consider the q-version of the Dirac - Hestenes equation of the form*

$$(\boldsymbol{D}_2^q\gamma^0\gamma^2\gamma^1 - m)\psi = 0, \tag{87}$$

where $\boldsymbol{D}_2^q$ is given by (41). Thus, their respective solution is given by:

$$\psi(x_2) = m\gamma^1\gamma^2\gamma^0\left\{\sum_{n=0}^{\infty}[\gamma_2 u(p_2)e^{iqp_2x_2}]^n + \sum_{n=0}^{\infty}[\gamma_2 u(p_2)e^{i(1-q)p_2x_2}]^n\right\} \tag{88}$$

.

Proof. First, we integrate over Ω_q in (87):

$$\begin{aligned}\int_{\Omega_q} \boldsymbol{D}_2^q \gamma^0\gamma^2\gamma^1\psi(\boldsymbol{v}_2(x_2))\mathbf{d}_q\boldsymbol{v}_2 &= m\int_{\Omega_q}\psi(\boldsymbol{v}_2(x_2))\mathbf{d}_q\boldsymbol{v}_2,\\ \gamma^2\psi(x_2)\gamma^0\gamma^2\gamma^1 &= m\int_{\Omega_q}\psi(\boldsymbol{v}_2(x_2))\mathbf{d}_q\boldsymbol{v}_2,\\ \gamma^2\psi(x_2)\gamma^0\gamma^2\gamma^1 &= \gamma^2 m\left\{\sum_{n=0}^{\infty}\left[\gamma^2\psi(qx_2)\right]^n+\sum_{n=0}^{\infty}[\gamma^2\psi((1-q)x_2)]^n\right\},\end{aligned}$$

If we assume the plane-wave ansatz $\psi(x_2)=u(p_2)e^{ip_2x_2}$, then we obtain:

$$\begin{aligned}\psi(x_2)\gamma^0\gamma^2\gamma^1 &= m\left\{\sum_{n=0}^{\infty}\left[\gamma^2u(p_2)e^{iqp_2x_2}\right]^n+\sum_{n=0}^{\infty}[\gamma^2u(p_2)e^{i(1-q)p_2x_2}]^n\right\},\\ \psi(x_2)\gamma^2\gamma^1 &= \gamma^0 m\left\{\sum_{n=0}^{\infty}\left[\gamma^2u(p_2)e^{iqp_2x_2}\right]^n+\sum_{n=0}^{\infty}[\gamma^2u(p_2)e^{i(1-q)p_2x_2}]^n\right\},\\ \psi(x_2)\gamma^1 &= -\gamma^2\gamma^0 m\left\{\sum_{n=0}^{\infty}\left[\gamma^2u(p_2)e^{iqp_2x_2}\right]^n+\sum_{n=0}^{\infty}[\gamma^2u(p_2)e^{i(1-q)p_2x_2}]^n\right\},\\ \psi(x_2) &= \gamma^1\gamma^2\gamma^0 m\left\{\sum_{n=0}^{\infty}\left[\gamma^2u(p_2)e^{iqp_2x_2}\right]^n+\sum_{n=0}^{\infty}[\gamma^2u(p_2)e^{i(1-q)p_2x_2}]^n\right\}.\end{aligned}$$

resulting finally (88). ☑

5. Suggestions for further work

There are further topics arising from this paper which are worth investigation., there is the problem of describing the function that depends on the quadratic variables $x^2, y^2, z^2, u^2, ux, uy, uz, xy$ and xz, this is $f(ux, uy, uz, xy, xz, yz, x^2, y^2, z^2, u^2)$ and introducing the q-quadratic difference operator over $f \in k\langle x, y, z, u\rangle$ [9]

$$\frac{\partial_{q^2}f}{\partial_{q^2}(ux)}=\frac{f(q^2u^2+ux)-f(ux)}{q^2u^2},$$

$$\frac{\partial_{q^2}f}{\partial_{q^2}(uy)}=\frac{f(q^2u^2+uy)-f(uy)}{q^2u^2},$$

$$\frac{\partial_{q^2}f}{\partial_{q^2}(uz)}=\frac{f(q^2u^2+zuz)-f(uz)}{q^2u^2},$$

$$\frac{\partial_{q^2}f}{\partial_{q^2}(yz)}=\frac{f(q^2y^2+yz)-f(yz)}{q^2y^2},$$

$$\frac{\partial_{q^2} f}{\partial_{q^2}(xy)} = \frac{f(q^2y^2 + xy) - f(xy)}{q^2y^2},$$
$$\frac{\partial_{q^2} f}{\partial_{q^2}(xz)} = \frac{f(q^2z^2 + xz) - f(xz)}{q^2z^2},$$

and for a function $f : \mathbb{R}^n \longrightarrow R$

$$\begin{aligned}
\frac{\partial_q f}{\partial_q x} &= \frac{f((x + q^2\mathrm{e}_0u)u) - f(xu)}{qu}, & \frac{\partial_q f}{\partial_q y} &= \frac{f((y + q^2\mathrm{e}_0u)u) - f(yu)}{qu},\\
\frac{\partial_q f}{\partial_q z} &= \frac{f(y(z + q^2\mathrm{e}_yy)) - f(yz)}{qy}, & \frac{\partial_q f}{\partial_q u} &= \frac{f((u - q^2\mathrm{e}_yy)y) - f(uy)}{qy},\\
\frac{\partial_q f}{\partial_q x} &= \frac{f((x + q^2\mathrm{e}_zz)z) - f(xz)}{qz}, & \frac{\partial_q f}{\partial_q z} &= \frac{f((z + q^2\mathrm{e}_0u)u) - f(zu)}{qu},\\
\frac{\partial_q f}{\partial_q y} &= \frac{f((y + q^2\mathrm{e}_xx)x) - f(yx)}{qx}, & \frac{\partial_q f}{\partial_q z} &= \frac{f((z + q^2\mathrm{e}_xx)x) - f(zx)}{qx}.
\end{aligned}$$

Analogously, the above expressions can be written in terms of spinor variables.

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