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On Effros continua and the uniform property of Effros

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Abstract. We recall a theorem by E. G. Effros about the actions of a separable complete metric group acting transitively on a complete metric space. We consider the definition, by D. P. Bellamy and K. F. Porter of an Effros continuum in the class of Hausdorff homogeneous continua. We also recall the definition of the uniform property of Effros for Hausdorff continua. We prove that a homogeneous Hausdorff continuum is an Effros continuum if and only if it has the uniform property of Effros. We consider the weak property of Effros introduced by F. W. Simmons and show that Hausdorff continua with the weak property of Effros are homogeneous. We introduce the uniform weak property of Effros. We show that it is equivalent to the definition given by Simmons and that a Hausdorff continuum with the uniform property of Effros has uniform the weak property of Effros.

Keywords: Continuum, decomposable continuum, Effros continuum, Hausdorff continuum, metric continuum, group action, indecomposable continuum, the uniform property of Effros, the weak uniform property of Effros, topological group.

MSC2020: 54B20, 54C60, 54F16.

Sobre los continuos de Effros y la propiedad uniforme de Effros

Resumen. Recordamos un teorema de E. G. Effros sobre las acciones de grupos métricos completos y separables actuando transitivamente en un espacio métrico completo. Consideramos la definición dada por D. P. Bellamy

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y K. F. Porter de un continuo de Effros en la clase de continuos de Hausdorff homogéneos. También recordamos la definición de la propiedad uniforme de Effros para continuos de Hausdorff. Demostramos que un continuo de Hausdorff homogéneo es un continuo de Effros si y sólo si dicho continuo tiene la propiedad uniforme de Effros. Consideramos la propiedad débil de Effros dada por F. W. Simmons y mostramos que continuos de Hausdorff con la propiedad débil de Effros son homogéneos. Introducimos la propiedad uniforme débil de Effros. Probamos que ésta es equivalente a la definición dada por Simmons y que un continuo de Hausdorff con la propiedad uniforme de Effros tiene la propiedad uniforme débil de Effros.

Palabras clave: Acción de grupo, continuo, continuo descomponible, continuo de Effros, continuo de Hausdorff, continuo indescomponible, continuo métrico, grupo topológico, propiedad uniforme de Effros, propiedad uniforme débil de Effros.

1. Introduction

In 1965, E. G. Effros proved a result about the action of a separable and complete metric group acting on a complete metric space using techniques of analysis [5, Theorem 2.1]. In 1987, F. D. Ancel presented a topological proof of Effros' theorem, he included the micro-transitivity action of the group on the complete metric space [1]. In 1991, D. P. Bellamy and K. F. Porter defined an Effros continuum and constructed a homogeneous Hausdorff continuum that is not an Effros continuum [2]. With the motivation to study homogeneous Hausdorff continua [10], [11], [13], we introduced the uniform property of Effros in [10]. We also introduced the set function $\wp$ [15] to extend to Hausdorff continua with the uniform property of Effros [15, Theorem 6.10] the Mutually Aposyndetic Decomposition of Homogeneous metric continua of J. R. Prajs [16, Theorem 3.1].

The paper is divided in four sections. After this introduction, section 1, we have section 2 which contains the necessary definitions for reading the paper. In section 3, we recall a result about the opennes of the evaluation map of the action of a complete metric group acting transitively on a metric space (Lemma 3.1). We also state a theorem by E. G. Effros about the actions of a separable complete metric group acting transitively on a complete metric space (Theorem 3.2). In section 4, we recall the definition of an Effros continuum given by D. P. Bellamy and K. F. Porter. We also recall the definition of the uniform property of Effros for Hausdorff continua. We prove that a homogeneous Hausdorff continuum is an Effros continuum if and only if it has the uniform property of Effros (Theorem 4.6). We present the definition of the weak property of Effros given by F. W. Simmons and showed that a Hausdorff continuum with the weak property of Effros is homogeneous (Theorem 4.7). We introduce the uniform version of the weak property of Effros, we show that it is equivalent to the definition given by Simmons (Theorem 4.8) and prove that a Hausdorff continuum with the uniform property of Effros has the uniform weak property of Effros (Theorem 4.9). We also show that if a continuum has the uniform weak property of Effros, then it is locally homogeneous according to Fora (Theorem 4.12).

2. Definitions

If Z is a Hausdorff topological space, given a subset A of Z, the interior of A is denoted by $Int(A)$, the boundary of A is denoted by $Bd(A)$, and the closure of A by $Cl(A)$.

Let Z be a Hausdorff space. If V and W are subsets of $Z \times Z$, then

$$-V = \{(z', z) \in Z \times Z \mid (z, z') \in V\}$$

and

$$V + W = \{(z, z'') \in Z \times Z \mid \text{there exists } z' \in Z \text{ such that} (z, z') \in V \text{ and } (z', z'') \in W\}.$$

We write $1V = V$ and for each positive integer n, $(n+1)V = nV + 1V$.

The diagonal of Z is the set $\Delta_Z = \{(z, z) \mid z \in Z\}$. An *entourage* of Δ_Z is a subset V of $Z \times Z$ such that $\Delta_Z \subset V$ and $V = -V$. The family of all entourages of the diagonal of Z is denoted by $\mathfrak{E}_Z$. If $V \in \mathfrak{E}_Z$ and $(z, z') \in V$, then we write $\rho_Z(z, z') < V$. If $V \in \mathfrak{E}_Z$ and $(z, z') \in (Z \times Z) \setminus V$, then we write $\rho_Z(z, z') \geq V$. If $V \in \mathfrak{E}_Z$ and $z \in Z$, then $B(z, V) = \{z' \in Z \mid \rho_Z(z, z') < V\}$. We have that if z, z' and z'' are points of Z, and V and W belong to $\mathfrak{E}_Z$ then the following hold [6, p. 426]:

(i) $\rho_Z(z, z) < V$.

(ii) $\rho_Z(z, z') < V$ if and only if $\rho_Z(z', z) < V$.

(iii) If $\rho_Z(z, z') < V$ and $\rho_Z(z', z'') < W$, then $\rho_Z(z, z'') < V + W$.

Let Z be a Tychonoff space. A *uniformity* on Z is a subfamily $\mathfrak{U}$ of $\mathfrak{E}_Z \setminus \{\Delta_Z\}$ such that:

(1) If $V \in \mathfrak{U}$, $W \in \mathfrak{E}_Z$ and $V \subset W$, then $W \in \mathfrak{U}$.

(2) If V and W belong to $\mathfrak{U}$, then $V \cap W \in \mathfrak{U}$.

(3) For every $V \in \mathfrak{U}$, there exists $W \in \mathfrak{U}$ such that $2W \subset V$.

(4) $\bigcap\{V \mid V \in \mathfrak{U}\} = \Delta_Z$.

A *uniform space* is a pair $(Z, \mathfrak{U})$ consisting of a nonempty set Z and a uniformity on the set Z. For any uniformity $\mathfrak{U}$ on a set Z, the family

$$\mathfrak{O} = \{G \subset Z \mid \text{for every } z \in G, \text{ there exists } V \in \mathfrak{U} \text{ such that } B(z, V) \subset G\}$$

is a topology on the set Z [6, 8.1.1]. The topology $\mathfrak{O}$ is called the *topology induced by the uniformity* $\mathfrak{U}$.

Remark 2.1. Note that, if the topology of Z is induced by a uniformity $\mathfrak{U}$ and $V \in \mathfrak{U}$, then, by [6, 8.1.3], $Int_Z(B(z, V))$ is an open neighbourhood of z.

Remark 2.2. Note that by [6, 8.3.13], for every compact Hausdorff space Z, there exists a unique uniformity $\mathfrak{U}_Z$ on Z that induces the original topology of Z.

Remark 2.3. Let Z be a Tychonoff space and let $\mathfrak{U}$ be a uniformity of Z that induces its topology. If $V \in \mathfrak{U}$, then we define the cover of Z, $\mathfrak{C}(V) = \{B(z, V) \mid z \in Z\}$.

We need the following result [6, 8.3.G]:

Theorem 2.4. *Let Z be a compact Hausdorff space and let $\mathfrak{U}_Z$ (Remark 2.2) be the unique uniformity of Z that induces its topology. Then for every open cover $\mathcal{W}$ of Z, there exists $V \in \mathfrak{U}_Z$ such that $\mathfrak{C}(V)$ refines $\mathcal{W}$.*

We say that a group G, which has a topology τ, is a *topological group* provided that the group operations are continuous; i.e., the functions given by:

$$\pi\colon G \times G \to G \qquad \pi((g_1, g_2)) = g_1 \cdot g_2$$

and

$$\xi\colon G \to G \qquad \xi(g) = g^{-1}$$

are continuous.

Let G be a group. If H and K are nonempty subsets of G, then define

$$H \cdot K = \{h \cdot k \mid h \in H \text{ and } k \in K\}$$

and

$$H^{-1} = \{h^{-1} \mid h \in H\}.$$

If $H = \{h\}$, then we write $h \cdot K$ instead of $\{h\} \cdot K$. Similarly, we write $H \cdot k$ instead of $H \cdot \{k\}$, when $K = \{k\}$.

An *action of a topological group G on a metric space X* is a map $\theta\colon G \times X \to X$ such that:

(1) $\theta(g, \theta(g', x)) = \theta(g \cdot g', x)$, where $\{g', g\} \subset G$ and for each $x \in X$

and

(2) $\theta(e_G, x) = x$, for each $x \in X$, where e_G is the identity element of G.

We use the following notation $\theta(g, x) = g \cdot x$.

Let G be a complete metric group acting on a metric space X. Then $G \cdot x$ is called the *orbit of x under the action of G on X*. Note that distinct orbits are disjoint. The set

$$\{G \cdot x \mid x \in X\}$$

of orbits is called the *orbit space determined by the action of G on X*, and it is denoted by X/G. We give X/G the quotient topology. The function $q\colon X \twoheadrightarrow X/G$ given by $q(x) = G \cdot x$ is the quotient map.

Let G be a topological group acting on a metric space X. We say that G *acts transitively on X*, if $G \cdot x = X$, for each $x \in X$. We say that G *acts micro-transitively on X* if for each $x \in X$ and each neighborhood U of e_G, $U \cdot x$ is a neighborhood of x in X.

Let G be a topological group acting on a metric space X. If $x \in X$, then

$$G_x = \{g \in G \mid g \cdot x = x\}$$

is a subgroup of G called the *stabilizer subgroup of* x. Note that G_x acts on G by multiplication on the right. The orbit space of this action, G/G_x, is the set

$$G/G_x = \{g \cdot G_x \mid g \in G\}.$$

Let G be a complete metric group acting on a metric space X. For each $x \in X$, define the map $\gamma_x : G \twoheadrightarrow X$ by $\gamma_x(g) = g \cdot x$, for every $g \in G$.

A *continuum* is a compact, connected Hausdorff space. A continuum X is *homogeneous* provided that for each pair of points x_1 and x_2 of X, there exists a homeomorphism $h : X \twoheadrightarrow X$ such that $h(x_1) = x_2$.

Regarding actions of topological groups on continua, a family of continuous actions on continua irreducible about a finite set and of type λ continua, such that every action has a fixed point may be found in [3].

3. A Theorem of E. G. Effros for metric spaces

In 1965, E. G. Effros proved a result about of a group action on a complete metric space using techniques of analysis [5, Theorem 2.1] (Theorem 3.2). In 1987, F. D. Ancel presented a topological proof of Effros' theorem, he included the micro-transitivity action of the group on the complete metric space [1]. We state, without proof, [1, Lemma 1 and Effros' Theorem]. Proofs of these results may be also found in [12, Lemma 4.2.9 and Theorem 4.2.25].

Lemma 3.1. *If G is a complete metric group acting transitively on a metric space X, then the following are equivalent:*

(a) *G acts micro-transitively on X;*

(b) *γ_x is an open map for each $x \in X$;*

(c) *γ_x is an open map for some $x \in X$.*

Let G be a complete metric group acting on a metric space X. If $x \in X$, define the function $\psi_x : G/G_x \to G \cdot x$ by $\psi_x(g \cdot G_x) = g \cdot x$.

Theorem 3.2. *If G is a separable complete metric group acting on a separable complete metric space X, then the following are equivalent:*

(a) *For each $x \in X$, the map $\psi_x : G/G_x \to G \cdot x$ is a homeomorphism.*

(b) *G acts micro-transitively on each orbit.*

(c) *Each orbit is of the second category (in itself).*

(d) *Each orbit is a G_δ subset of X.*

(e) *X/G is a T_0 space.*

If X is a continuum, then $\mathcal{H}(X)$ denotes the family of homeomorphisms of X onto itself with the compact-open topology. Also, 1_X denotes the identity map on X. Based on [18], C. L. Hagopian noted that considering the action of $\mathcal{H}(X)$ on X, Theorem 3.2 may be used to study homogeneous metric continua and he applied it to show that every decomposable subcontinuum of a homogeneous indecomposable plane continuum contains a homogeneous indecomposable subcontinuum [8, Theorem 1]. Later, he proved that each subcontinuum of an indecomposable homogeneous plane continuum is indecomposable [9, Theorem 1]. In order to show [8, Theorem 1], he proved a slightly different version [8, Lemma 4] of Theorem 3.3. A proof of this result may be found in [12, Theorem 4.2.31].

Theorem 3.3. *If X is a homogeneous continuum, with metric d, then for each $\varepsilon > 0$, there exists $\delta > 0$ such that if x and y belong to X and $d(x, y) < \delta$, then there exists $h \in \mathcal{H}(X)$, such that $h(x) = y$ and $d(z, h(z)) < \varepsilon$, for every $z \in X$.*

Theorem 3.3 is now known as the Effros' Theorem in the theory of homogeneous metric continua. Also, Theorem 3.3 motivated the following definition:

A metric space X has the *property of Effros* provided that for each $\varepsilon > 0$, there exists $\delta > 0$ such that if x and y are points of X and $d(x, y) < \delta$, then there exists $h \in \mathcal{H}(X)$ such that $h(x) = y$ and $d(z, h(z)) < \varepsilon$, for every $z \in X$. The number δ is called an *Effros number for the given* ε. A homeomorphism $h \in \mathcal{H}(X)$ satisfying that $d(z, h(z)) < \varepsilon$ for each $z \in X$ is called an *ε-homeomorphism.*

Remark 3.4. Observe that the euclidean space of dimension n, $\mathbb{R}^n$, satisfies the property of Effros.

Note the following result, we prove the Hausdorff version of it in Theorem 4.3.

Theorem 3.5. *If X is a metric continuum satisfying the property of Effros, then X is homogeneous.*

As a consequence of Theorems 3.3 and 3.5, we have:

Corollary 3.6. *A metric continuum X is a homogeneous continuum if and only if X has the property of Effros.*

4. Effors continua

Based on Lemma 3.1, D. P. Bellamy and K. F. Porter defined an *Effros continuum* in the following way:

A homogeneous continuum X is an *Effros continuum* provided that for each point x of X, the evaluation map $\gamma_x \colon \mathcal{H}(X) \twoheadrightarrow X$ given by $\gamma_x(h) = h(x)$ is open, where $\mathcal{H}(X)$ denotes the family of homeomorphisms of X onto itself with the compact-open topology.

They gave an example of a homogeneous continuum that is not an Effros continuum [2]. This means that the theory of homogeneous continua is different from the theory of homogeneous metric continua.

Remark 4.1. It is known that if X is a continuum and $\mathcal{H}(X)$ is the group of homeomorphisms of X, then $\mathcal{H}(X)$ is a topological group with the compact-open topology [6, p. 441].

With the motivation to study homogeneous continua [10], [11], [13], we introduced the *uniform property of Effros* in [10] as follows:

A continuum X has the *uniform property of Effros*, provided that for each $U \in \mathfrak{U}_X$ (Remark 2.2), there exists $V \in \mathfrak{U}_X$ such that if x_1 and x_2 are two points of X with $\rho_X(x_1, x_2) < V$, then there exists a homeomorphism $h\colon X \twoheadrightarrow X$ such that $h(x_1) = x_2$ and $\rho_X(x, h(x)) < U$, for all elements x of X. The entourage V is called an *Effros entourage for* U. A homeomorphism $h\colon X \twoheadrightarrow X$ satisfying $\rho_X(x, h(x)) < U$, for each point x of X, is called a U*-homeomorphism*.

A comparison of the uniform property of Effros with several definitions of local homogeneity may be found in [14].

Next, we present the uniformity to the homeomorphism group of a continuum.

Notation. Let X be a continuum. For every $U \in \mathfrak{U}_X$ (Remark 2.2), let $\widehat{U}$ be the entourage of the diagonal $\Delta_{\mathcal{H}(X)} \subset \mathcal{H}(X) \times \mathcal{H}(X)$ given by:

$$\widehat{U} = \{(g_1, g_2) \in \mathcal{H}(X) \times \mathcal{H}(X) \mid \rho_X(g_1(x), g_2(x)) < U, \text{ for all } x \in X\}.$$

It is known that the family $\widehat{\mathcal{U}} = \{\widehat{U} \mid U \in \mathfrak{U}_X\}$ is a base for a uniformity on $\mathcal{H}(X)$; the uniformity generated by this family is called the *uniformity of uniform convergence* induced by $\mathfrak{U}_X$ and is denoted by $\widehat{\mathfrak{U}_X}$ [6, p. 440].

We need a special case of [6, 8.2.7]:

Theorem 4.2. *Let X be a continuum. Then the topology on $\mathcal{H}(X)$ induced by the uniformity $\widehat{\mathfrak{U}_X}$ of uniform convergence coincides with the compact-open topology on $\mathcal{H}(X)$, and depends only on the topology of X.*

The next result says that each continuum with the uniform property of Effros is homogeneous.

Theorem 4.3. *If X is a continuum having the uniform property of Effros, then X is homogeneous.*

Proof. Let x_1 and x_2 be two points of X and let $U \in \mathfrak{U}_X$ be such that $\rho_X(x_1, x_2) \geq U$. Since X has the uniform property of Effros, there exists an Effros entourage V for U. Let $V' \in \mathfrak{U}_X$ be such that $2V' \subset V$. Note that $\{Int(B(x, V')) \mid x \in X\}$ is an open cover of X. Since X is connected, by [4, Theorem (2F2)], there exist $w_1, \ldots, w_n$ in X such that $x_1 \in Int(B(w_1, V'))$, $x_2 \in Int(B(w_n, V'))$ and $Int(B(w_j, V')) \cap Int(B(w_k, V')) \neq \emptyset$ if and only if $|j-k| \leq 1$. Let $z_0 = x_1$ and let $z_{n+1} = x_2$. For each $j \in \{1, \ldots, n-1\}$, let $z_j \in Int(B(w_j, V')) \cap Int(B(w_{j+1}, V'))$. Let $j \in \{0, \ldots, n\}$. Since $\rho(z_j, z_{j+1}) < V$, there exists a U-homeomorphism $h_j\colon Z \twoheadrightarrow Z$ such that $h_j(z_j) = z_{j+1}$. Let $h = h_n \circ \cdots \circ h_0$. Then $h\colon X \twoheadrightarrow X$ is a homeomorphism such that $h(x_1) = x_2$. Therefore, X is a homogeneous space. ☑

The next result is useful for the proof of Theorem 4.6.

Theorem 4.4. *If X is a continuum with the uniform property of Effros, then for each element x of X and every $U \in \mathfrak{U}_X$, there exists an open subset $O_{U,x}$ with $z \in O_{U,x}$ such that if $x' \in O_{U,x}$, there exists a U-homeomorphism $h\colon X \twoheadrightarrow X$ such that $h(x) = x'$.*

Proof. Let x be a point of X and let $U \in \mathfrak{U}_X$. Since X has the uniform property of Effros, there exists an Effros entourage V for U. Note that $Int(B(x, V))$ is an open subset of X containing x (Remark 2.1). Let $x' \in Int(B(x, V))$. Then $\rho_X(x, x') < V$. Thus, there exists a U-homeomorphism $h\colon X \twoheadrightarrow X$ such that $h(x) = x'$. ☑

The next theorem answers the following question, asked in the *Seminario de Topología "Rafael Isaacs"* of the Universidad Industrial de Santander:

Question 4.5. If X is a continuum with the uniform property of Effros, is it true that X is an Effros continuum?

Theorem 4.6. *Let X be a continuum. Then X is an Effros continuum if and only if X has the uniform property of Effros.*

Proof. Suppose X is an Effros continuum. Then, by definition, X is homogeneous. By Theorem 4.2, the topology induced by $\widehat{\mathfrak{U}_X}$ on $\mathcal{H}(X)$ coincides with the compact-open topology. Also, $\widehat{\mathcal{U}}$ is a base for the uniformity $\widehat{\mathfrak{U}_X}$. Let x_0 be a point of X and let $U \in \mathfrak{U}_X$ (Remark 2.2). Let $U' \in \mathfrak{U}_X$ be such that $2U' \subset U$. Since X is an Effros continuum, the map γ_{x_0} is an open map. Then $\gamma_{x_0}(Int(B(1_X, \widehat{U'})))$ is an open neighbourhood of x_0 in X. Then there exists $V_{x_0} \in \mathfrak{U}_X$ such that $B(x_0, V_{x_0}) \subset \gamma_{x_0}(Int(B(1_X, \widehat{U'})))$. Thus, if $x \in B(x_0, V_{x_0})$, then there exists $h \in Int(B(1_X, \widehat{U'}))$ such that $h(x_0) = x$ and $\rho_X(z, h(z)) < U'$, for every $z \in X$. Observe that $\{Int(B(x, V_x)) \mid x \in X\}$ is an open cover of X (Remark 2.1). Since X is compact, by Theorem 2.4, there exists $V \in \mathfrak{U}_X$ such that $\mathfrak{C}(V)$ refines $\{Int(B(x, V_x)) \mid x \in X\}$. Let x_1 and x_2 be points of X such that $\rho_X(x_1, x_2) < V$. Then there exists x_3 in X such that x_1 and x_2 both belong to $Int(B(x_3, V_{x_3}))$. Let h_1 and h_2 be elements of $Int(B(1_X, \widehat{U'}))$ such that $h_1(x_3) = x_1$ and $h_2(x_3) = x_2$. Let $h = h_2 \circ h_1^{-1}$. Then h is a homeomorphism of X and $h(x_1) = x_2$. Let z be a point of X. Since $\rho_X(z, h(z)) = \rho_X(z, h_2 \circ h_1^{-1}(z))$, $\rho_X(z, h_1^{-1}(z)) = \rho_X(h_1 \circ h_1^{-1}(z), h_1^{-1}(z)) < U'$, $\rho_X(h_1^{-1}(z), h_2 \circ h_1^{-1}(z)) < U'$ and $2U' \subset U$, we obtain that $\rho_X(z, h(z)) < U$. Therefore, h is a U-homeomorphism and X has the uniform property of Effros.

Suppose X is a continuum with the uniform property of Effros. By Theorem 4.3, X is a homogeneous continuum. Let x be a point of X, let $\mathcal{O}$ be an open subset of $\mathcal{H}(X)$ and let $x' \in \gamma_x(\mathcal{O})$. Then there exists $h \in \mathcal{O}$ such that $\gamma_x(h) = h(x) = x'$. Since $\mathcal{O}$ is an open subset of $\mathcal{H}(X)$, there exists $U \in \mathfrak{U}_X$ such that $h \in B(h, \widehat{U}) \subset \mathcal{O}$ (see **Notation** above). By Theorem 4.4, there exists an open subset W of X such that $x \in W$ and if $z \in W$, then there exists a U-homeomorphism of X with $h(x') = z$. Thus, $\rho_{\mathcal{H}(X)}(e, h) < \widehat{U}$. We show that $W \subset \gamma_x(\mathcal{O})$. To this end, let $w \in W$. Then there exists a U-homeomorphism f such that $f(x') = w$. Since f is a U-homeomorphism, we have that $\rho_{\mathcal{H}(X)}(e, f) < \widehat{U}$. Hence, $\rho_{\mathcal{H}(X)}(h, f \circ h) < \widehat{U}$. Thus, $f \circ h \in B(h, \widehat{U})$ and $f \circ h(x) = f(h(x)) = f(x') = w$. Hence, $w \in \gamma_x(B(h, \widehat{U})) \subset \gamma_x(\mathcal{O})$. Therefore, $\gamma_x(\mathcal{O})$ is an open subset of X and γ_x is an open map. ☑

F. W. Simmons defined the *weak property of Effros* to study homogeneous continua separated by some pair of points. In fact, he showed that if X is a homogeneous continuum which has the weak property of Effros and is separated by some pair of points, then it is separated by each pair of its points; i.e., X is a Hausdorff circle [17, Theorem 12]. Simmons defined the *weak property of Effros* in [17] as follows:

A continuum X has the *weak property of Effros* provided that for each pair of points p and q of X and every open subset A_q of X having q, there exists an open subset B_p of X containing p such that for each element x of B_p, there exists a homeomorphism $h_{p,x} \colon X \twoheadrightarrow X$ such that $h_{p,x}(p) = x$ and $h_{p,x}(q) \in A_q$.

Next result extends Theorem 4.3 to continua with the weak property of Effros.

Theorem 4.7. *If X is a continuum with the weak Effros property, then X is a homogeneous space.*

Proof. Let x_1 and x_2 be two points of X and let x_3 be a point in $X \setminus \{x_1, x_2\}$. Let A_3 be an open subset of X having x_3. Since X has the weak property of Effros, for each $x \in X \setminus \{x_3\}$, there exists an open subset B_x of X containing x such that for each $x' \in B_x$, there exists a homeomorphism $h_{x,x'} \colon X \twoheadrightarrow X$ such that $h_{x,x'}(x) = x'$ and $h_{x,x'}(x_3) \in A_3$. Also, for B_{x_1}, there exists an open subset B_{x_3}, having x_3 such that for every $x' \in B_{x_3}$, there exists a homeomorphism $h_{x_3,x'} \colon X \twoheadrightarrow X$ such that $h_{x_3,x'}(x_3) = x'$ and $h_{x_3,x'}(x_1) \in B_{x_1}$. Note that $\{B_x \mid x \in X\}$ is an open cover of X. Since X is connected, by [4, (2F2)], there exist $w_1, \ldots, w_n$ in X such that $x_1 \in B_{w_1}$, $x_2 \in B_{w_n}$ and $B_{w_j} \cap V_{w_k} \neq \emptyset$ if and only if $|j - k| \leq 1$. Let $z_0 = x_1$ and let $z_n = x_2$. For each $j \in \{1, \ldots, n-1\}$, let $z_j \in B_{w_j} \cap B_{w_{j+1}}$. Consider w_1. By hypothesis, there exist two homeomorphisms $h_{w_1,z_0}, h_{w_1,z_1} \colon X \twoheadrightarrow X$ such that $h_{w_1,z_0}(w_1) = z_0$ and $h_{w_1,z_1}(w_1) = z_1$. Let $h_1 \colon X \twoheadrightarrow X$ be given by $h_1 = h_{w_1,z_1} \circ h^{-1}_{w_1,z_0}$. Then h_1 is a homeomorphism and $h_1(z_0) = z_1$. If $j \in \{2, \ldots, n\}$, by hypothesis, there exist two homeomorphisms $h_{w_j,z_j}, h_{w_j,z_{j-1}} \colon X \twoheadrightarrow X$ such that $h_{w_j,z_{j-1}}(w_j) = z_{j-1}$ and $h_{w_j,z_j}(w_j) = z_j$. Let $h_j \colon X \twoheadrightarrow X$ be given by $h_j = h_{w_j,z_j} \circ h^{-1}_{w_j,z_{j-1}}$. Then h_j is a homeomorphism and $h_j(z_{j-1}) = z_j$. Let $h = h_n \circ \cdots \circ h_1$. Then $h \colon X \twoheadrightarrow X$ is a homeomorphism such that $h(x_1) = x_2$. Therefore, X is a homogeneous continuum. ☑

We present a uniform version of the weak property of Effros:

A continuum X has the *uniform weak property of Effros* if for every pair of points p and q of X and each $U_q \in \mathfrak{U}_X$ (Remark 2.2), there exists $V_p \in \mathfrak{U}_X$ such that if $x \in X$ satisfies that $\rho_X(p, x) < V_p$, then there exists a homeomorphism $h_{p,x} \colon X \twoheadrightarrow X$ such that $h_{p,x}(p) = x$ and $\rho_X(q, h_{p,x}(q)) < U_q$. The entourage V_p is called a *uniform weak Effros entourage for U_q*.

Theorem 4.8. *A continuum X has the weak property of Effros if and only if X has the uniform weak property of Effros.*

Proof. Suppose X has the weak property of Effros. Let p and q be two points of X and let $U_q \in \mathfrak{U}_X$ (Remark 2.2). Then $Int(B(q, U_q))$ is an open subset of X having q (Remark 2.1). Hence, there exists an open subset B_p of X containing p such that if $x \in B_p$, then there exists a homeomorphism $h_{p,x} \colon X \twoheadrightarrow X$ such that $h_{p,x}(p) = x$ and $h_{p,x}(q) \in Int(B(q, U_q))$. Let $V_p \in \mathfrak{U}_X$ be such that $B(p, V_p) \subset B_p$. Thus, if $\rho_X(p, z) < V_p$, then $z \in B_p$ and there exists a homeomorphism $h_{p,z} \colon X \twoheadrightarrow X$ such that $h_{p,z}(p) = z$ and $h_{p,z}(q) \in Int(B(q, U_q))$. Hence, $\rho_X(q, h_{p,z}(q)) < U_q$. Therefore, X has the uniform weak property of Effros.

Assume that X has the uniform weak property of Effros. Let p and q be two points of X and let A_q be an open subset of X having q. Let $U_q \in \mathfrak{U}_X$ be such that $B(q, U_q) \subset A_q$. Since X has the uniform weak property of Effros, there exists $V_p \in \mathfrak{U}_X$ such that if $x \in X$ satisfies that $\rho_X(p, x) < V_p$, then there exists a homeomorphism $h_{p,x} \colon X \twoheadrightarrow X$ such that $h_{p,x}(p) = x$ and $\rho_X(q, h_{p,x}(q)) < U_q$. Note that $Int(B(p, V_p))$ is an open subset of X with $p \in Int(B(p, V_p))$ (Remark 2.1). Let $z \in Int(B(p, V_p))$. Then $\rho_X(p, z) < V_p$. Hence, there exists a homeomorphism $h_{p,z} \colon X \twoheadrightarrow X$ such that $h_{p,z}(p) = x$ and $\rho_X(q, h_{p,z}(q)) < U_q$. Since $B(q, U_q) \subset A_q$, we obtain that $h_{p,z}(q) \in A_q$. Therefore, X has the weak property of Effros. ☑

The following theorem shows that each continuum with the uniform property of Effros has the uniform weak property of Effros.

Theorem 4.9. *If X is a continuum with the uniform property of Effros, then X has the uniform weak property of Effros.*

Proof. Let p and q be two elements of X and let $U_q \in \mathfrak{U}_X$ (Remark 2.2). Since X has the uniform property of Effros, there exists $V_p \in \mathfrak{U}_X$ such that if x_1 and x_2 belong to X and $\rho_X(x_1, x_2) < V_p$, then there exists U_q-homeomorphism $h \colon X \twoheadrightarrow X$ such that $h(x_1) = x_2$. Hence, if $x \in X$ and $\rho_X(p, x) < V_p$, then there exists a U_q-homeomorphism $h \colon X \twoheadrightarrow X$ such that $h(p) = x$. Since h is a U_q-homeomorphism, we have that $\rho_X(q, h(q)) < U_q$. Therefore, X has the uniform weak property of Effros. ☑

Corollary 4.10. *If X is a continuum with the uniform property of Effros, then X has the weak property of Effros.*

A continuum X is *locally homogeneous$_F$*, *according to Fora* [7], provided that for each point x of X, there exists an open subset B_x of X containing x such that for every element x' of B_x, there exists a homeomorphism $h_{x,x'} \colon X \twoheadrightarrow X$ such that $h_{x,x'}(x) = x'$.

With a similar argument to the one given in the proof of Theorem 4.3, we have:

Theorem 4.11. *If X a continuum is locally homogeneous$_F$, then X is homogeneous.*

Theorem 4.12. *If X is a continuum that has the uniform weak property of Effros, then X is locally homogeneous$_F$.*

Proof. Let x and z be points of X and let $U_z \in \mathfrak{U}_X$ (Remark 2.2). Since X has the uniform weak property of Effros, there exists a uniform weak Effros entourage V_x for U_z. Note that $Int(B(x, V_x))$ is an open subset of X having x (Remark 2.1). Hence, if $x' \in Int(B(x, V_x))$, we have that $\rho_X(x, x') < V_x$. Thus, since X has the uniform weak property of Effros, there exists a homeomorphism $h_{x,x'} \colon X \twoheadrightarrow X$ such that $h_{x,x'}(x) = x'$. Therefore, X is locally homogeneous$_F$. ☑

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